as plasma phosphates decline, it can be assumed that liver phosphate concentration increases. Zerbe (1979) provided measurements of plasma phosphate (mg/dl) for 33 participants (20 obese, 13 control) after a glucose challenge. Figure 2 depicts group means for eight measurement occasions.

An appropriate model for Zerbe's (1979) data may be the segmented linear spline with two phases. In such a model, the first phase (here, phosphate depletion) is fit with a linear model, and the second phase (recovery) is fit with a different linear model. The point at which one line changes into the next (here, when depletion turns into recovery) is commonly called the *knot*, *joint*, *change point*, or *transition point*. Often the knot is known in advance; other times it is treated as an estimated parameter. Additionally, the knot is not always expected to be the same for all individuals. Recognizing this, Cudeck and Klebe (2002) fit a segmented spline to Zerbe's data using multilevel modeling, showing that the knot may be treated as a fixed quantity, an estimated parameter, or even a random coefficient. Random knots are relatively novel, but have been modeled successfully using traditional (Cudeck, 1996; Cudeck & du Toit, 2003) and Bayesian (Dominicus, Ripatti, Pedersen, & Palmgren, 2008; McArdle & Wang, 2008; Muniz Terrera, van den Hout, & Matthews, 2011; Wang & McArdle, 2008) mixed effects models and growth mixture models (Kohli, 2011; Kohli, Harring, & Hancock, 2013; Li, Duncan, Duncan, & Hops, 2001). Here, we use the LGM framework and Steps 1 through 4 to reparameterize the traditional segmented spline. In the LGM context, knots have been treated as fixed, known quantities (Bollen & Curran, 2006; Flora, 2008) or as estimated parameters (Harring, Cudeck, & du Toit, 2006; Kohli et al., 2013), but not as random coefficients. Clearly, it would be beneficial to treat knots as randomly varying across individuals to more accurately mirror individual differences in the timing of phosphate rebound, and in many other contexts as well. Additionally, the ability to specify random knots in SEM makes available many of the advantages of latent variable modeling.

We begin with the traditional mixed-model expression of a two-segment linear spline target function:

$$y = \begin{cases} \theta_1 + \theta_2 t & t \leq \theta_\kappa \\ \theta_3 + \theta_4 t & t > \theta_\kappa \end{cases} \tag{10}$$

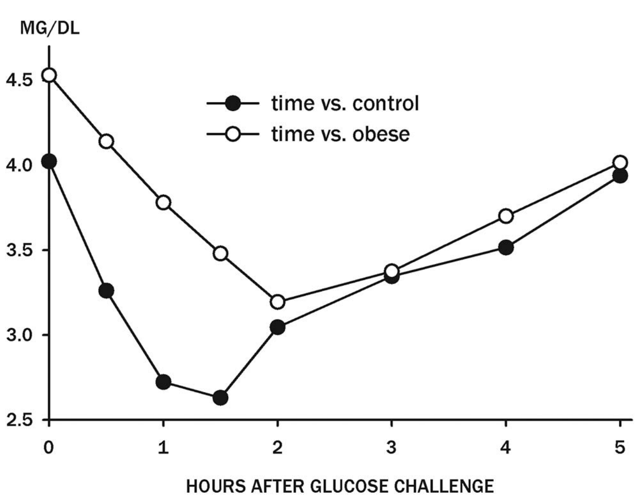

Figure 2. Obese and normal-weight group means for plasma phosphate concentration at eight unequally spaced occasions.

where θ_1 and θ_2 are the intercept and slope for the first segment, θ_3 and θ_4 are the intercept and slope for the second segment, and θ_κ is the knot. The segments are assumed to join at the knot, so there are effectively only four growth coefficients rather than the apparent five; that is, once any four of the θs are known, the fifth is determined. Cudeck and Klebe (2002) approached the problem from a mixed model perspective, treating the four free parameters as random effects, with separate means for the two groups but a common Level 2 covariance matrix. We would like to treat the knot as a random coefficient using LGM. However, it is not possible to specify this parameterization of the target function in SEM directly in a way that permits treating the knot as a random coefficient.

The first step will be to reparameterize the model, but the purpose of doing so is to render the model in such a way that it can be linearized and fit using SEM while preserving θ_κ as a parameter. Harring et al. (2006), Kohli and Harring (2013), and Kohli et al. (2013) reparameterized the target function in Equation 10 in a way that makes it possible to specify in SEM as a partially nonlinear model.[7] That is, the unknown change point θ_κ can be estimated, but must be treated as a fixed rather than random coefficient. Their parameterization[8] is

$$y = f(\boldsymbol{\theta}, t) = \omega_1 + \omega_2 t + \omega_3 \sqrt{(t - \theta_\kappa)^2}. \tag{11}$$

(Note that if we suspect $\theta_2 > \theta_4$, we would subtract rather than add the ω_3 term.)

In the now reparameterized Equation 11, the three ω coefficients are functions of the original growth coefficients θ_1 through θ_4. They still represent aspects of change (respectively, the average intercept across segments, the average slope across segments, and half the difference between the two slopes).[9] However, we are more concerned with the knot. It is important to note that θ_κ survived the reparameterization intact, and bears the same interpretation as in the original parameterization in Equation 10. Crucially, Equation 11 is in a form that can be linearized and modeled using SEM, whereas Equation 10 is not.

The second step is to linearize the reparameterized target function to express it in a form that will be more palatable to SEM software. Anticipating that all parameters in the target function will be treated as random coefficients, the first partial derivatives of Equation 11 with respect to each coefficient, evaluated at the population point, are

$$\begin{aligned} \partial f / \partial \omega_1 &= 1 \\ \partial f / \partial \omega_2 &= t \\ \partial f / \partial \omega_3 &= \sqrt{(t - \mu_\kappa)^2} \\ \partial f / \partial \theta_\kappa &= \mu_3 (\mu_\kappa - t) / \sqrt{(t - \mu_\kappa)^2} \end{aligned}$$

Thus, the linearized target function is

[7] A partially (or conditionally) nonlinear model is one in which only those parameters that enter the model linearly may be treated as random coefficients (Blozis, 2012; Blozis & Cudeck, 1999).

[8] A similar but slightly different model was used for estimating random knots in the mixed effects modeling framework by Crockett, Harvey, Guo, Francis, and Brouwers (2005).

[9] We can express the original model coefficients in terms of these new ones: $\theta_1 = \omega_1 + \omega_3\theta_\kappa$, $\theta_2 = \omega_2 - \omega_3$, $\theta_3 = \omega_1 - \omega_3\theta_\kappa$, $\theta_4 = \omega_2 + \omega_3$, and $\theta_\kappa = \theta_\kappa$. See Harring et al. (2006) and Kohli and Harring (2013) for more complete details, and see online supplemental Appendix 8 for the derivation of Equation 11.

$$\begin{aligned}\tilde{y} &= f(\boldsymbol{\theta},t) + (\omega_1-\mu_1)\frac{\partial f}{\partial \omega_1}\bigg|_{\mu_1} + (\omega_2-\mu_2)\frac{\partial f}{\partial \omega_2}\bigg|_{\mu_2} + (\omega_3-\mu_3)\frac{\partial f}{\partial \omega_3}\bigg|_{\mu_3} + (\theta_\kappa-\mu_\kappa)\frac{\partial f}{\partial \theta_\kappa}\bigg|_{\mu_\kappa}\\ &= \left[\mu_1+\mu_2 t+\mu_3\sqrt{(t-\mu_\kappa)^2}\right] + (\omega_1-\mu_1)(1) + (\omega_2-\mu_2)t\\ &\quad + (\omega_3-\mu_3)\sqrt{(t-\mu_\kappa)^2} + (\theta_\kappa-\mu_\kappa)\mu_3(\mu_\kappa-t)/\sqrt{(t-\mu_\kappa)^2}\end{aligned}$$

In the third step, we use the derivatives obtained in Step 2 as factor loadings,[10] substituting values of $t = 0, \ldots, T$:

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 & \sqrt{(0-\mu_\kappa)^2} & \mu_3(\mu_\kappa-0)/\sqrt{(0-\mu_\kappa)^2} \\ 1 & 1 & \sqrt{(1-\mu_\kappa)^2} & \mu_3(\mu_\kappa-1)/\sqrt{(1-\mu_\kappa)^2} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & T & \sqrt{(T-\mu_\kappa)^2} & \mu_3(\mu_\kappa-T)/\sqrt{(T-\mu_\kappa)^2} \end{bmatrix}.$$

The linearized model can then be expressed in matrix form as

$$\mathbf{y}_j = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta}_j + \boldsymbol{\varepsilon}_j, \tag{12}$$

where $\boldsymbol{\tau}$, the intercept vector, represents the target model evaluated at the population point, and $\boldsymbol{\Lambda}\boldsymbol{\eta}_j$ represents the deviation of individual j's trajectory from the mean implied by $\boldsymbol{\tau}$. The vector $\boldsymbol{\varepsilon}_j$ contains occasion-specific errors for individual j; for simplicity, we assume $\boldsymbol{\varepsilon}_{ij} \sim \mathrm{N}(0, \sigma^2_\varepsilon)$ with time-homoscedastic variance.

We make use of our modification of SLCM to specify the model in Equation 12. Recall that in the modified method, $\boldsymbol{\alpha} = \mathbf{0}$ and can be omitted, and $\boldsymbol{\tau}$ is set equal to the desired mean trajectory at the parameter estimates. The model is depicted graphically in Figure 3, Panel A, and is an *unconditional* model that does not consider obesity (we will include obesity next in a conditional model, shown in Panel B). Symbols for elements of the random coefficient covariance matrix are omitted from the figure for simplicity, but can be represented as

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_1 & & & \\ \psi_{2,1} & \psi_2 & & \\ \psi_{3,1} & \psi_{3,2} & \psi_3 & \\ \psi_{\kappa,1} & \psi_{\kappa,2} & \psi_{\kappa,3} & \psi_\kappa \end{bmatrix}.$$

Fitting the model to data using Mplus 7.2 (L. K. Muthén & Muthén, 1998–2014) yields

$$\begin{aligned}\hat{\mu}_1 &= 3.400\ (.128)\\ \hat{\mu}_2 &= -.258\ (.034)\\ \hat{\mu}_3 &= .539\ (.040),\\ \hat{\mu}_\kappa &= 1.588\ (.112)\end{aligned}$$

and

$$\hat{\boldsymbol{\Psi}} = \begin{bmatrix} .457\ (.135) & & & \\ -.030\ (.028) & .013\ (.010) & & \\ -.024\ (.030) & -.009\ (.009) & .027\ (.013) & \\ -.008\ (.085) & .018\ (.026) & .001\ (.027) & .254\ (.110) \end{bmatrix},$$

with $\boldsymbol{\varepsilon}_{ij} \sim \mathrm{N}[0, .103(.013)]$.[11] The model fits poorly ($\chi^2_{29} = 68.163$, $p < .001$; root mean square error of approximation [RMSEA] = .202, 90% CI [.140, .265]; non-normed fit index [NNFI] = .849), so the results of this example should be understood only in a didactic spirit and should not be taken to reflect on the substantive domain of plasma phosphate dynamics. The mean knot is estimated to occur 1.588 hr after the glucose challenge, with a standard deviation of approximately one half hour ($\hat{\psi}_\kappa = .254$). That is, even though phosphate levels begin to rebound on average just after the 1.5-hr mark, there exists considerable individual variability around this mean.

Regarding this individual variability, recall that the goal of this example was not only to show how the knot may be treated as a random coefficient within the SEM/LGM framework but also to predict individual differences in the knot using obesity status as a Level 2 predictor. The structural equation for the random coefficients $\boldsymbol{\eta}_j$ in the *conditional* model may be written as

$$\boldsymbol{\eta}_j = \boldsymbol{\Gamma}\mathbf{x}_j + \boldsymbol{\zeta}_j$$

where $\boldsymbol{\Gamma}$ contains structural coefficients (γs) linking growth factors to exogenous measured variables in $\mathbf{x}_j$. Regressing the growth factors on obesity status yields

$$\begin{aligned}\hat{\mu}_1 &= 3.553\ (.160)\\ \hat{\mu}_2 &= -.212\ (.042)\\ \hat{\mu}_3 &= .490\ (.050),\\ \hat{\mu}_\kappa &= 1.951\ (.123)\end{aligned} \tag{13}$$

$$\hat{\boldsymbol{\Psi}} = \begin{bmatrix} .421\ (.126) & & & \\ -.041\ (.027) & .010\ (.009) & & \\ -.013\ (.028) & -.006\ (.008) & .024\ (.013) & \\ -.084\ (.077) & -.001\ (.023) & .011\ (.023) & .076\ (.082) \end{bmatrix},$$

and

[10] The Taylor series expansion used in SLCM and our modification of SLCM requires the target function to be differentiable with respect to its parameters in the neighborhood of the point of expansion. However, the derivatives of Equation 11 with respect to ω_3 and θ_κ are undefined where $t = \mu_\kappa$. The range in which most subjects' random knots are expected to fall may be inferred using $\hat{\boldsymbol{\psi}}_\kappa$ (e.g., $\hat{\mu}_\kappa \pm 1.96\hat{\boldsymbol{\psi}}_\kappa^{1/2}$). When such a range includes a known singularity, such as those occurring at the points of measurement in the current model, then the quality of the Taylor series expansion may become compromised and inference regarding parameter estimates tenuous.

[11] We used bootstrapping to obtain standard errors because we encountered unstable estimation in both the unconditional and conditional models arising from the small sample size; standard errors are based on 2,367 converged solutions out of 3,000 requested for the unconditional model, and on 2,105 out of 3,000 for the conditional model.

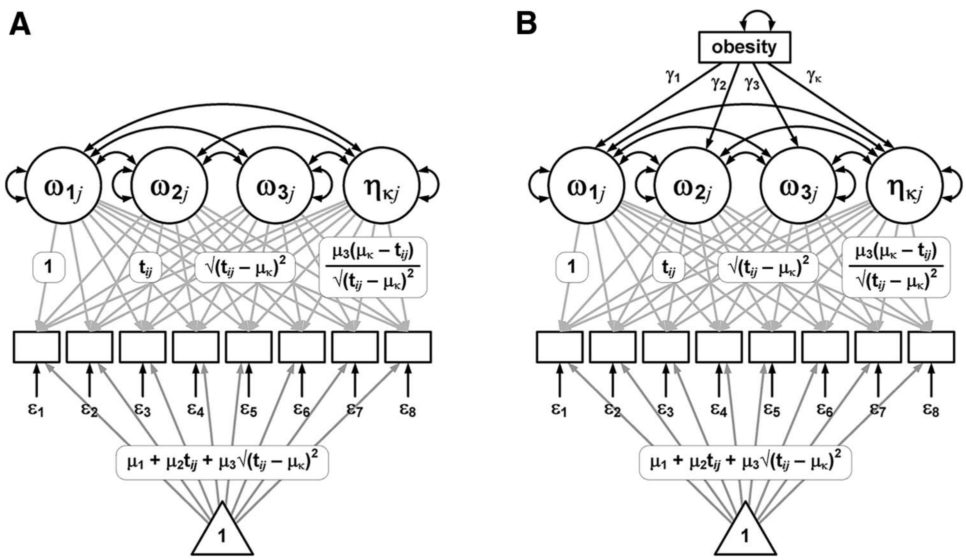

Figure 3. (A) A two-segment linear spline model for plasma phosphate depletion and recovery. The mean knot is represented by μ_κ, whereas the variance of the $\eta_{\kappa j}$ factor represents variability in the knot. (B) The same model is represented with obesity as a Level 2 predictor of growth coefficients. Because obesity is coded (0 = obese, 1 = normal), μ_κ represents the model-implied knot for the obese group, and the coefficient γ_κ represents the group mean difference (in units of hours) between normal-weight and obese groups.

$$\hat{\boldsymbol{\Gamma}} = \begin{bmatrix} -.387\ (.254) \\ -.117\ (.067) \\ .124\ (.079) \\ -1.013\ (.253) \end{bmatrix},$$

with $\boldsymbol{\varepsilon}_{ij} \sim N[0, .103(.013)]$. The model fits poorly ($\chi^2_{33} = 70.711$, $p < .001$; RMSEA = .186, 90% CI [.126, .246], NNFI = .849), so the results of this example again should be treated only didactically. Because the obese group was coded "0" and the normal weight group was coded "1," the means in Equation 13 reflect model-implied values of the growth coefficients for the obese group. The negative effect of obesity status on the knot latent variable conveys the finding that normal weight individuals' mean knot occurred approximately an hour before the obese group's mean knot ($\hat{\gamma}_4 = -1.013$ hr), a result that tallies well with the means plotted in Figure 2. Mplus syntax is provided in the online supplemental appendix to reproduce results for both the unconditional and conditional models.[12]

Using these reparameterized models, we were able to predict individual differences in the timing of a metabolic event: the point in time at which plasma phosphates cease to be absorbed by the liver and begin to be returned to the blood. The advantages of specifying the segmented linear spline model with a random knot in SEM are clear.

Example 2: Average Rate of Change (ARC) and Half-Life

Our second example is drawn from a procedural learning study concerning verbal and quantitative skill acquisition, using data originally collected by Scott Chaiken of Armstrong Laboratory, Brooks Air Force Base (for a thorough description of these data, see Blozis, 2004).[13] The dependent measures are verbal and quantitative accuracy scores, consisting of aggregated response times within 12 trial blocks. Individual and mean-level data based on 228 cases are illustrated in Figure 4.

In this example, we reparameterize a common learning function to contain random coefficients for two relatively novel parameters, one reflecting an absolute learning rate and another reflecting scale-free learning rate. The first of these is the ARC—an individual's mean instantaneous linear slope across the full span of a trajectory (Kelley, 2009; Kelley & Maxwell, 2008). This mean derivative is simply $\Delta y_i/\Delta t_i$, regardless of the functional form of change, but as Kelley and Maxwell (2008) emphasized, the ARC is distinct from the mean slope in all but the simplest models of change. In repeated measures scenarios, every individual has his or her own ARC. Thus, it would be useful to treat the ARC as a random coefficient.[14] Kelley and Maxwell did not specify the ARC as a parameter or coefficient of the model, but they did suggest that the model's functional form may be considered by fitting a functional form of change and using predicted values of y to obtain the ARC. This would avoid ignoring intermediate occasions and increase accuracy when estimating the ARC.

The second parameter of interest is the *half-life*, defined as the amount of time necessary for a function to cover half the distance from any point to its asymptote (Rausch, 2004, 2008; Willett, 1989). Certain exponential models are characterized by a constant half-life parameter. As with the ARC, each individual can have his or her own half-life parameter. We suggest that the half-life can be a useful metric of learning rate. It reflects not the final level or

[12] See online supplemental Appendices 2 and 3 for Mplus syntax, and see Appendices 10 and 11 for annotated output.

[13] These data are included in installations of LISREL.

[14] Models involving ARC parameters are similar to, but distinct from, the *growth rate models* discussed by Zhang, McArdle, and Nesselroade (2012), which involve reparameterizing a target growth function in terms of a first derivative at a given occasion, such that the derivative (growth rate) can be treated as a model parameter or random coefficient. The ARC, in contrast, is a parameter representing the average of these growth rates over a given span of time.

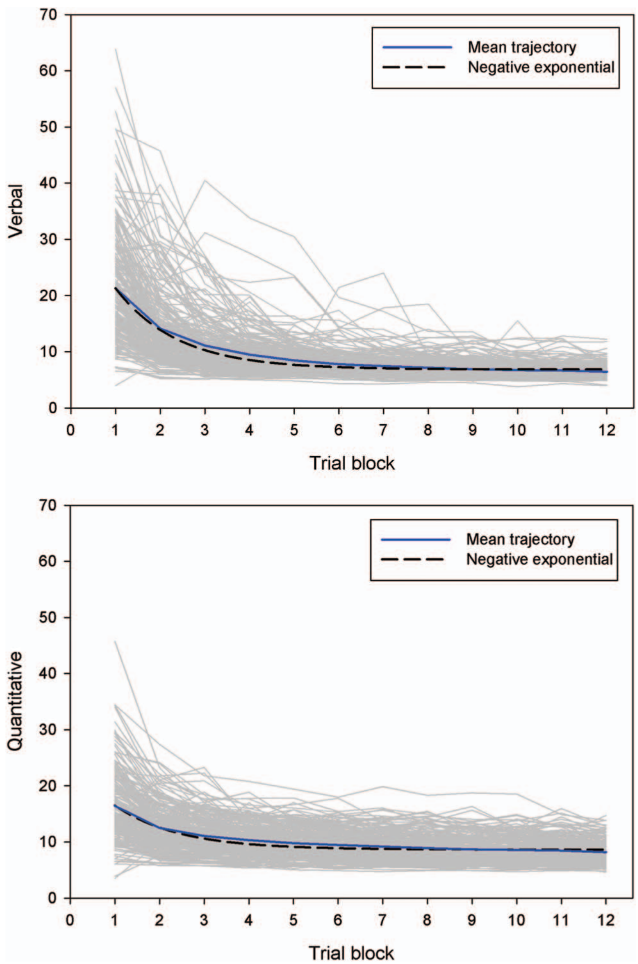

Figure 4. Verbal and quantitative skill acquisition response times from Chaiken's data. See the online article for the color version of this figure.

overall amount of learning, but rather the rapidity with which it is approached. Because half-life parameters are on a common metric (time), they can be used to compare learning rates for variables in different domains, even if they are measured on different scales.

Our first task was to choose an appropriate functional form to describe change in Chaiken's skill acquisition data. We used a three-parameter negative exponential (NE) function previously used by Blozis (2004) with the same data:

$$y = \theta_1 - (\theta_1 - \theta_2)\exp(-\theta_3[t-1]). \tag{14}$$

Here, θ_1 is the horizontal asymptote, θ_2 is the model-implied value of y at the initial trial, and θ_3 governs the rate of change. The rate parameter θ_3 governs how quickly the curve falls toward its horizontal asymptote at $y = \theta_1$. Importantly, although the NE function contains a parameter reflecting rate of change, it does not contain parameters that directly describe the ARC or half-life. Thus, our second task was to reparameterize the model such that the ARC and half-life are treated as random coefficients.

To reparameterize the NE function to contain a half-life parameter, we must find a way to express the half-life in terms of existing model parameters. We want the time at which the proportion of total change equals 1/2. Assuming negative growth (i.e., $\theta_2 > \theta_1$), the half-life will be

$$\begin{aligned}\frac{1}{2} &= \frac{\theta_2 - \left[\theta_1 - (\theta_1 - \theta_2)\exp(-\theta_3(\theta_h - 1))\right]}{\theta_2 - \theta_1} \\ &= 1 - \exp(-\theta_3(\theta_h - 1)).\end{aligned}$$

That is, θ_h is the value of t for which the distance between the initial value of y (θ_2) and the value of y at θ_h is exactly one half the total distance between θ_2 and the asymptote θ_1. Expressing the rate parameter as a function of θ_h yields

$$\theta_3 = \frac{\ln(½)}{1 - \theta_h}.$$

Substitution into the target function yields

$$\begin{aligned}y &= \theta_1 - (\theta_1 - \theta_2)\exp\left(-\frac{\ln(½)}{1-\theta_h}(t-1)\right) \\ &= \theta_1 - (\theta_1 - \theta_2)(½)^{\frac{t-1}{\theta_h - 1}}.\end{aligned} \tag{15}$$

This half-life parameterization can be further parameterized to trade the initial value parameter θ_2 for an ARC parameter. The ARC is formally defined as (Kelley & Maxwell, 2008)

$$\mathrm{ARC} = \frac{1}{t_T - t_1}\int_{t_1}^{t_T} f'(y)dt = \frac{y_{t=t_T} - y_{t=t_1}}{t_T - t_1}.$$

That is, the ARC is the difference between the model-implied values of y at the beginning and end of the desired span of time, divided by that span. Substituting the model expression for the two instances of y yields

$$\begin{aligned}\theta_{\mathrm{ARC}} &= \frac{\theta_1 - (\theta_1 - \theta_2)(½)^{\frac{t_T - 1}{\theta_h - 1}} - \left(\theta_1 - (\theta_1 - \theta_2)(½)^{\frac{t_1 - 1}{\theta_h - 1}}\right)}{t_T - t_1} \\ &= \frac{(\theta_2 - \theta_1)\left((½)^{\frac{t_T - 1}{\theta_h - 1}} - (½)^{\frac{t_1 - 1}{\theta_h - 1}}\right)}{t_T - t_1}.\end{aligned}$$

Therefore,

$$\theta_2 = \frac{\theta_{\mathrm{ARC}}(t_T - t_1)}{(½)^{\frac{t_T - 1}{\theta_h - 1}} - (½)^{\frac{t_1 - 1}{\theta_h - 1}}} + \theta_1.$$

Substitution into Equation 15 yields the fully reparameterized target function:

$$\begin{aligned}y &= f(\boldsymbol{\theta}, t) \\ &= \theta_1 - (\theta_1 - \theta_2)(½)^{\frac{t-1}{\theta_h - 1}} \\ &= \theta_1 + \frac{\theta_{\mathrm{ARC}}(t_T - t_1)(½)^{\frac{t-1}{\theta_h - 1}}}{(½)^{\frac{t_T - 1}{\theta_h - 1}} - (½)^{\frac{t_1 - 1}{\theta_h - 1}}}.\end{aligned} \tag{16}$$

The third step is to linearize Equation 16. We theorize that each individual may have a different half-life and a different ARC, reflecting individual differences in learning rates. It is also natural to expect individual differences in asymptote. Thus, all three parameters of the reparameterized target function will be treated as random coefficients. Moreover, we wish to model both verbal and quantitative skill acquisition simultaneously in a bivariate trajectory model, which requires asymptote, half-life, and ARC coefficients for each domain.

The derivatives of Equation 16 with respect to each coefficient, evaluated at the population point, are

$$\frac{\partial f}{\partial \theta_1} = 1,$$

$$\frac{\partial f}{\partial \theta_{\mathrm{ARC}}} = \frac{(t_T - t_1)(½)^{\frac{t-1}{\theta_h - 1}}}{(½)^{\frac{t_T-1}{\theta_h-1}} - (½)^{\frac{t_1-1}{\theta_h-1}}}, \text{ and}$$

$$\frac{\partial f}{\partial \theta_h} = \frac{(½)^{\frac{t}{\theta_h-1}}\theta_h\left((½)^{\frac{t_1}{\theta_h-1}}(t-t_1) + (½)^{\frac{t_T}{\theta_h-1}}(t_T - t)\right)(t_T - t_1)\ln(½)}{(\theta_h - 1)^2\left((½)^{\frac{t_T-1}{\theta_h-1}} - (½)^{\frac{t_1-1}{\theta_h-1}}\right)^2}.$$

Two sets of these derivatives exist—one for verbal and one for quantitative.

In the next step, these derivatives are used as loadings in the 24×4 block diagonal matrix $\mathbf{\Lambda}$ (with parameters associated with V and Q superscripted as such),

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}^{\mathrm{V}} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}^{\mathrm{Q}} \end{bmatrix},$$

with columns of $\mathbf{\Lambda}^{\mathrm{V}}$ (12×2) containing $\partial f/\partial\eta_1^{\mathrm{v}}$, $\partial f/\partial\eta_{\mathrm{ARC}}^{\mathrm{v}}$ and $\partial f/\partial\eta_h^{\mathrm{v}}$ columns of $\mathbf{\Lambda}^{\mathrm{Q}}$ (12×2) containing $\partial f/\partial\eta_1^{\mathrm{Q}}$, $\partial f/\partial\eta_{\mathrm{ARC}}^{\mathrm{Q}}$, and $\partial f/\partial\eta_h^{\mathrm{Q}}$. Values of $t = 1, \ldots, 12$ are substituted within both $\mathbf{\Lambda}^{\mathrm{V}}$ and $\mathbf{\Lambda}^{\mathrm{Q}}$. The linearized model can then be expressed in the form of Equation 9, where $\mathbf{\Lambda\eta}_j$ represents the deviation of individual j's verbal and quantitative trajectories from the respective mean trajectories implied by $\boldsymbol{\tau}$. The vector $\boldsymbol{\varepsilon}_j$ again contains occasion-specific errors for individual j. Errors for each outcome will have homoscedastic variances over time and autoregressive covariance structures determined by parameters ρ^{V} and ρ^{Q}, as in Blozis (2004).

We next specify the linearized model, the diagram for which is depicted in Figure 5. The asymptote, half-life, and ARC parameters for verbal and quantitative domains are estimated as

$$\begin{aligned} \hat\theta_1^{\mathrm{V}} &= 6.845\ (.100) & \hat\theta_1^{\mathrm{Q}} &= 8.629\ (.143) \\ \hat\theta_{\mathrm{ARC}}^{\mathrm{V}} &= -1.312\ (.058) & \hat\theta_{\mathrm{ARC}}^{\mathrm{Q}} &= -.715\ (.031)\,. \\ \hat\theta_h^{\mathrm{V}} &= 1.962\ (.026) & \hat\theta_h^{\mathrm{Q}} &= 1.993\ (.033) \end{aligned}$$

The ARC parameters indicate statistically significant negative average rates of change for each domain, whereas the half-life parameters indicate that the time that elapses for typical individuals to reach a point halfway between their current status and their learning level is comparable for the verbal and quantitative tasks. A Wald test of $(\theta_h^{\mathrm{V}} - \theta_h^{\mathrm{Q}})$ indicated no statistically significant difference in the domain-specific half-life parameters ($p = .454$).

The random coefficient covariance matrix is

$$\hat{\mathbf{\Psi}} = \begin{bmatrix} \hat\psi_{1(\mathrm{V})} \\ \hat\psi_{\mathrm{ARC(V)},1(\mathrm{V})} & \hat\psi_{\mathrm{ARC(V)}} \\ \hat\psi_{h(\mathrm{V}),1(\mathrm{V})} & \hat\psi_{h(\mathrm{V}),\mathrm{ARC(V)}} & \hat\psi_{h(\mathrm{V})} \\ \hat\psi_{1(\mathrm{Q}),1(\mathrm{V})} & \hat\psi_{1(\mathrm{Q}),\mathrm{ARC(V)}} & \hat\psi_{1(\mathrm{Q}),h(\mathrm{V})} & \hat\psi_{1(\mathrm{Q})} \\ \hat\psi_{\mathrm{ARC(Q)},1(\mathrm{V})} & \hat\psi_{\mathrm{ARC(Q)},\mathrm{ARC(V)}} & \hat\psi_{\mathrm{ARC(Q)},h(\mathrm{V})} & \hat\psi_{\mathrm{ARC(Q)},1(\mathrm{Q})} & \hat\psi_{\mathrm{ARC(Q)}} \\ \hat\psi_{h(\mathrm{Q}),1(\mathrm{V})} & \hat\psi_{h(\mathrm{Q}),\mathrm{ARC(V)}} & \hat\psi_{h(\mathrm{Q}),h(\mathrm{V})} & \hat\psi_{h(\mathrm{Q}),1(\mathrm{Q})} & \hat\psi_{h(\mathrm{Q}),\mathrm{ARC(Q)}} & \hat\psi_{h(\mathrm{Q})} \end{bmatrix}$$

$$= \begin{bmatrix} 1.652(.207) \\ -.193(.084) & .712(.069) \\ -.131(.051) & .025(.030) & .216(.023) \\ 1.664(.235) & -.597(.126) & -.183(.071) & 4.187(.423) \\ -.035(.046) & .122(.028) & .015(.016) & -.293(.068) & .208(.021) \\ -.037(.025) & -.021(.014) & .016(.008) & -.064(.036) & -.031(.008) & .046(.006) \end{bmatrix}$$

All of the random coefficient variances and most of the covariances exceed chance levels, indicating that the coefficients reflecting learning rate vary significantly from person to person, and often covary with each other and with the final level of learning (asymptote). The verbal and quantitative ARC coefficients were positively related, as were the verbal and quantitative half-life coefficients. The model does not fit well by conventional criteria ($\chi^2_{293} = 2435.191$, $p < .001$, RMSEA $= .179$, 90% CI [.173, .186], NNFI $= .745$), so results should be taken in a didactic spirit and do not reflect on the substantive domain of verbal and quantitative skill acquisition. Mplus syntax is provided in the online supplemental appendix to reproduce these results.[15]

Recall that Browne (1993) showed that if no restrictions are placed on the random effect covariance matrix, reparameterization

[15] See online supplemental Appendices 5 and 13 for Mplus syntax and annotated output, respectively.

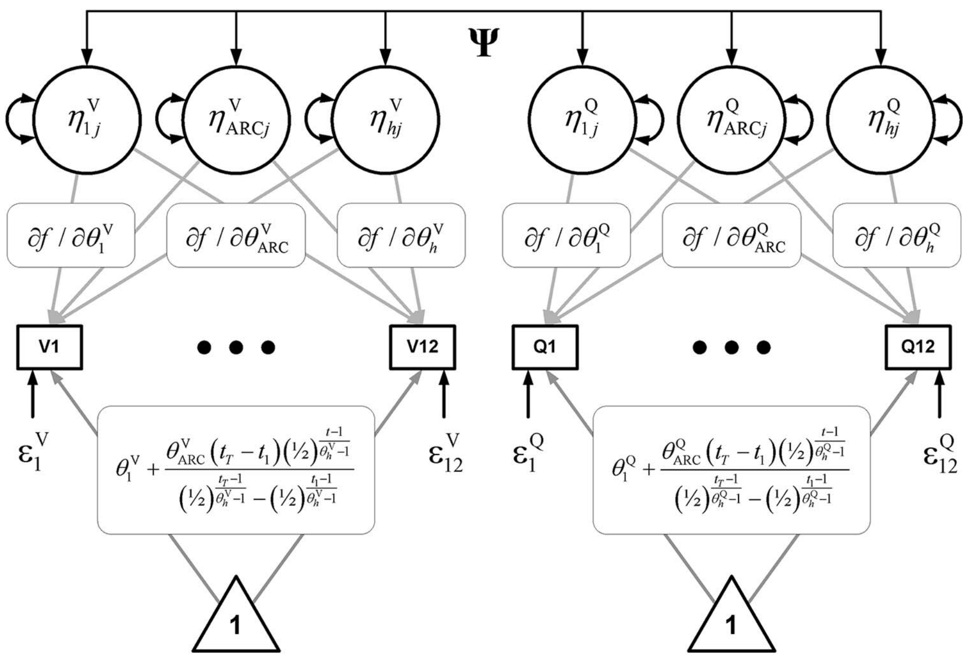

Figure 5. A bivariate trajectory model for verbal (V) and quantitative (Q) skill acquisition, in which each trajectory is a negative exponential curve with asymptote (θ_1), average rate of change (θ_{ARC}), and half-life (θ_h) random coefficients. Not depicted are the occasion-specific errors, which have an autoregressive covariance structure within each domain.

yields an equivalent model. For completeness, and to illustrate the equivalence of the reparameterized target function to the original target function in Equation 14, we also fit a model using the original parameterization. Following all the preceding steps, but with this other target function in mind, we obtain identical model fit (χ^2_{293} = 2435.191, $p < .001$, RMSEA = .179, 90% CI [.173, .186], NNFI = .745). Estimates of the target function parameters are

$$\begin{array}{lllllll}\hat{\theta}_1^{\mathrm{V}} & = & 6.845(.100) & \quad & \hat{\theta}_1^{\mathrm{Q}} & = & 8.629(.143)\\ \hat{\theta}_2^{\mathrm{V}} & = & 21.278(.661) & & \hat{\theta}_2^{\mathrm{Q}} & = & 16.492(.404)\\ \hat{\theta}_3^{\mathrm{V}} & = & .721(.020) & & \hat{\theta}_3^{\mathrm{Q}} & = & .698(.023)\end{array}.$$

These are the estimated horizontal asymptotes, intercepts, and rate of change parameters for verbal and quantitative skill acquisition. The random coefficient covariance matrix is

$$\hat{\boldsymbol{\Psi}} = \begin{bmatrix} \hat{\psi}_{1(\mathrm{V})} & & & & & \\ \hat{\psi}_{2(\mathrm{V}),1(\mathrm{V})} & \hat{\psi}_{2(\mathrm{V})} & & & & \\ \hat{\psi}_{3(\mathrm{V}),1(\mathrm{V})} & \hat{\psi}_{3(\mathrm{V}),2(\mathrm{V})} & \hat{\psi}_{3(\mathrm{V})} & & & \\ \hat{\psi}_{1(\mathrm{Q}),1(\mathrm{V})} & \hat{\psi}_{1(\mathrm{Q}),2(\mathrm{V})} & \hat{\psi}_{1(\mathrm{Q}),3(\mathrm{V})} & \hat{\psi}_{1(\mathrm{Q})} & & \\ \hat{\psi}_{2(\mathrm{Q}),1(\mathrm{V})} & \hat{\psi}_{2(\mathrm{Q}),2(\mathrm{V})} & \hat{\psi}_{2(\mathrm{Q}),3(\mathrm{V})} & \hat{\psi}_{2(\mathrm{Q}),1(\mathrm{Q})} & \hat{\psi}_{2(\mathrm{Q})} & \\ \hat{\psi}_{3(\mathrm{Q}),1(\mathrm{V})} & \hat{\psi}_{3(\mathrm{Q}),2(\mathrm{V})} & \hat{\psi}_{3(\mathrm{Q}),3(\mathrm{V})} & \hat{\psi}_{3(\mathrm{Q}),1(\mathrm{Q})} & \hat{\psi}_{3(\mathrm{Q}),2(\mathrm{Q})} & \hat{\psi}_{3(\mathrm{Q})} \end{bmatrix}$$

$$= \begin{bmatrix} 1.652(.207) & & & & & \\ 3.783(.976) & 92.144(8.816) & & & & \\ -.147(.059) & -.468(.391) & .272(.044) & & & \\ 1.664(.235) & 8.242(1.468) & -.205(.082) & 4.187(.423) & & \\ 2.051(.606) & 23.425(4.200) & -.392(.232) & 7.414(.985) & 35.773(3.456) & \\ -.072(.049) & .371(.318) & .036(.018) & -.126(.073) & .540(.204) & .179(.030) \end{bmatrix}$$

Note that the horizontal asymptote parameter estimates are the same across parameterizations; this parameter was preserved in the reparameterization process. Mplus syntax is provided in the online supplemental appendix to reproduce these results.[16]

Discussion

We had three goals in this article. First, we aimed to provide theoretical background for Preacher and Hancock's (2012) four-step method for reparameterizing and fitting linear and nonlinear growth curve models, and elaborated on it. Second, we described a modification to the traditional structured latent curve model that enables fitting a broader class of nonlinear growth models and renders model specification easier. Third, we presented two examples illustrating the specification of several aspects of change as random coefficients, some new to the SEM literature. We further supplied extensive Mplus syntax in an online supplemental appendix to make the application of such models more feasible for applied researchers. Our core message is that growth curve models may be considerably more flexible than most researchers suspect, and that strategic reparameterization is one practical way to unlock that flexibility.

Good model specification requires a model's parameters to have direct, substantive interpretations that are meaningful within the research context. Many models used by social scientists are already parameterized such that their parameters are readily interpretable. For example, multiple regression models contain, by default, intercepts and linear slopes that bear directly on questions of scientific interest. Item response models contain parameters that correspond directly to item difficulty and discrimination, which are of direct interest to researchers and test constructors. However, many models, despite providing accurate descriptive matches to observed data, do not contain directly interpretable parameters. In such cases, more interesting parameters must be derived through often-cumbersome second-stage analyses, and treating the new parameters as random coefficients is beyond most researchers' abilities. Still other models contain parameters that are interpretable, yet not as interesting as other quantifiable aspects of the model might be.

In such cases, we advocate strategic reparameterization as a general approach to model specification. Learning the basic approach to reparameterization will enable researchers to think creatively about how they specify models to directly test hypotheses of scientific interest and substantive importance. In this article, we described how to reparameterize a growth model such that aspects of change are represented as fixed values, as estimated parameters, or as random coefficients, and how to fit such reparameterized models within the SEM framework. In the process, we also described a slight modification of the standard SLCM method that facilitates model specification for many nonlinear models, and, in some cases, makes it possible to specify models that were not specifiable under standard SLCM.

We illustrated these steps using two examples involving empirical data. The first involved fitting a reparameterized spline model to a sample of obese and normal-weight individuals, treating the knot point as a predictable random coefficient. The second involved fitting parallel NE growth curves to verbal and quantitative skill acquisition data, treating the ARC and half-life as random coefficients.

When to Use Reparameterization

We noted that reparameterization enables the researcher to treat an aspect of change as a known, fixed constant. We did not illustrate this capability, but it could be used to render a model more parsimonious by incorporating known information. For example, if the knot point in our first example had been known a priori and did not vary over individuals, it could be fixed to that known value. Using reparameterization to treat an aspect of change as an estimated parameter or random coefficient, on the other hand, enables the researcher to investigate whether the aspect of change is predicted (moderated) by person-level predictors. We see this as the most important use of reparameterization, although there are others, as we mentioned earlier. These include (a) *convenience*—it is often more straightforward to estimate a parameter directly, and therefore to obtain a standard error and CI for it, than to compute it post hoc as a function of parameter estimates; (b) *stability*—if one parameterization has difficulty converging, sometimes another parameterization will not; and (c) *imposing constraints*—parameterizing a model such that only certain values are permissible is a way of constraining its range.

We have found reparameterization to be particularly useful in models of growth or change. In such models it is useful to consider (a) those aspects of change that are missing from the model but which we would like to treat as parameters or random coefficients, given the opportunity; and (b) which existing parameters we could live without. Usually it will be possible to sacrifice the latter for the former through reparameterization. For example, models of learning processes often contain an intercept parameter and one or more parameters linked to change over time. However, the intercept parameter (as in ordinary regression) is rarely of substantive interest. Rate of growth or change is of key interest. Additional parameters of interest in the learning context might include the total amount learned over the course of the study, or how long it takes to learn a subject to some criterion amount. Parameters or random coefficients reflecting the level of learning at the end of a study, or the time it takes an individual to reach a certain criterion of learning, would seem to be of far greater interest to education researchers than parameters related to students' starting points. In this article, we examined two examples of parameterizing novel aspects of change: treating the knot point in a segmented spline model as a random coefficient, and treating two measures of learning rate as random coefficients. But the basic idea of strategic reparameterization extends easily to other settings.

Comments on the Modeling Framework

A key distinction that should be borne in mind is the one between *marginal* or *population-average* (PA) and *subject-specific* (SS) models (Davidian & Giltinan, 1995, 2003; Demidenko, 2004; Lindstrom & Bates, 1990; Serroyen, Molenberghs, Verbeke, & Davidian, 2009; Zeger, Liang, & Albert, 1988). All applications of SLCM are PA models in the sense that only the mean trend is required to follow the functional form specified in the target function. This is ensured by the explicit requirement that $E[\mathbf{y}_j] = \mathbf{f}(\boldsymbol{\theta},\mathbf{t})$. Individual differences are accommodated by the inclusion

[16] See online supplemental Appendices 4 and 12 for Mplus syntax and annotated output, respectively.

of random effects for some or all aspects of change. However, PA models do not require individuals to follow the same functional form as the means, although model-implied subject-specific trajectories often will resemble the mean trend.[17] Thus, PA models tend to be used in situations in which the primary questions are about the mean trend. SS models, in contrast, require individuals to follow the target function, but make no such requirement for the mean trend. In SS models (termed *random-effects models* by Serroyen et al., 2009), the means do not necessarily follow a (simple) known functional form, although it is possible to aggregate the individual trajectories to get an idea of how the means change over time. Thus, SS models tend to be used when the primary focus is on modeling individuals' response trajectories. By and large, SLCM models (and our modification) are PA models, not SS models (Blozis, 2007a; Blozis, Conger, & Harring, 2007; Cudeck & Harring, 2007; Harring, 2009). However, when the model is linear in the random effects, the model is both PA and SS (Serroyen et al., 2009; Vonesh & Chinchilli, 1997). This property has been termed *dynamic consistency* (Singer & Willett, 2003; Vandergrift, 2004; Willett, 1989). Dynamically consistent models include linear and polynomial growth models and, more generally, *partially nonlinear* or *conditionally nonlinear* models, in which the intrinsically nonlinear parameters are treated as fixed coefficients but the linear parameters may vary across persons. We refer the reader to lucid discussions of the distinctions between PA and SS models by Cudeck and Harring (2007), Davidian and Giltinan (1995, 2003), Lindstrom and Bates (1990), and Vonesh and Chinchilli (1997).

Error Covariance Structures

We have been largely silent on how the error covariance structure is specified. With few exceptions (e.g., Blozis, 2004), the SLCM literature typically assumes errors to be independent over time (Blozis, 2007a; Blozis et al., 2007), more out of a desire for simplicity, not necessity. It has been repeatedly argued and shown that misspecification of the Level 1 error covariance structure translates into biased random effect parameters and biased standard errors elsewhere in the model (Grimm & Widaman, 2010; Gurka, Edwards, & Muller, 2011; Harring & Blozis, 2014). Cudeck and Harring (2007, p. 623) described the list of options for the error covariance structure as "truly dizzying." Guidance in parameterizing error covariance structures can be found in our Example 2 syntax, as well as in Rovine and Molenaar (1998, 2000), Blozis and Cudeck (1999), Harring and Blozis (2014), and Davidian and Giltinan (1995, Section 4.2).

Comments on Software Implementation

There are two points at which specialized software is relevant to our discussion. First, Steps 1 and 2 often will involve recourse to calculus. Many users will find it convenient to use software capable of symbolic calculus, such as Maple or Mathematica. We used the latter, and provide some example code in the online supplemental appendix.[18] Calculus is commonly used at two points. First, such software can be a great help in reparameterizing the target function. Reparameterizations that involve locating the maximum, minimum, or point of greatest change require locating points at which derivatives equal zero or themselves reach a maximum or minimum; in these cases, calculus is essential. Second, once the reparameterized target function has been identified, linearizing the function requires computing partial derivatives with respect to the function's coefficients. These actions are possible to do by hand, but the process can be quite tedious and error-prone.

The second point at which specialized software becomes indispensable is the model-fitting stage. Not all SEM software is capable of imposing the constraints required as part of the SLCM approach and our modification of it. We have found Mplus particularly useful in this regard, although LISREL, Mx, OpenMx, SAS PROCs CALIS and TCALIS, and lavaan all have constraint capabilities that vary in flexibility. Grimm and Ram (2009) provided code for a number of nonlinear models in Mplus and SAS, Blozis (2007a) provided code in LISREL and Mx, and Grimm et al. (2010) provided code for SLCM mixture models in Mplus and OpenMx. We include Mplus syntax for each of our examples, including the motivating inverse quadratic function example. Similar models can be fit using nonlinear mixed effects modeling. Nonlinear extensions of multilevel (random coefficients) modeling are very powerful and flexible when the goal is to estimate interindividual differences in intraindividual change that follows a nonlinear trajectory. Such models are commonly implemented using SAS PROC NLMIXED (e.g., Grimm & Ram, 2009), the R package lme4, OpenBUGS, Mx, and, to a limited extent, the MULTILEV module of LISREL.

Conclusion

We end by urging psychological researchers to make greater use of nonlinear trajectory models. Nonlinear models have a long history in psychology (see McArdle & Nesselroade, 2003, for an overview of early uses in psychology), but seem to have fallen somewhat out of fashion despite being more appropriate than linear growth curves in most situations (Cudeck & Harring, 2007). In our opinion, with rare exceptions, nonlinear trajectory models remain remarkably underutilized relative to their clear potential for psychological research. In other fields, a rich variety of nonlinear functions have been proposed to describe diverse phenomena. Gompertz (1825) proposed a sigmoidal function to describe the relationship between mortality and age (Winsor, 1932). Nonlinear functions potentially useful for the social sciences can be drawn from a surprising variety of sources, including biology (Karkach, 2006), botany (Hunt, 1982), oncology (Tabatabai, Williams, & Bursac, 2005), forestry (Leech & Ferguson, 1981; Sweda, 1984; Yang, Kozak, & Smith, 1978), and pharmacokinetics (Davidian & Giltinan, 1995, 2003). Many examples can be seen in treatments by Bates and Watts (1988), Davidian and Giltinan (1995), Huet, Bouvier, Poursat, and Jolivet (2004), Leech and Ferguson (1981), Mead and Pike (1975), Nelder (1966), Pinheiro and Bates (2000), Ratkowsky (1983, 1990), Seber and Wild (1989), Singer and Willett (2003), Sit and Poulin-Costello (1994), and Vonesh and Chinchilli (1997).

[17] Because random effects are treated as latent variables in this context, it is possible to estimate individual trajectories by obtaining factor scores and explicitly including them in Equation 12.

[18] See online supplemental Appendix 6 for Mathematica code for obtaining partial derivatives of an exponential function.

To date, only a few nonlinear functions have been modeled using SLCM, and only one or two parameterizations of each have been attempted. These are the *logistic curve* (Browne, 1993), various *exponential curves* (Blozis, 2004, 2007b; Blozis, Harring, & Mels, 2008; Browne, 1993; Grimm, Ram, & Hamagami, 2011), the *Gompertz curve* (Browne, 1993; Browne & du Toit, 1991), a modified *two-parameter logistic curve* (Blozis, 2007a), the *Michaelis-Menten curve* (Harring, Kohli, Silverman, & Speece, 2012), the *Preece-Baines model* (Grimm et al., 2011), and the *negative exponential curve* (Ghisletta, Kennedy, Rodrigue, Lindenberger, & Raz, 2010). In fact, there are several kinds of logistic and exponential curves, and potentially many different ways to parameterize each of these curves. Further development and application of SLCM and related methodology offers clear potential for both applied and methodological work. We hope our exposition of these issues will make researchers in psychology more willing, and more able, to employ nonlinear models in their own work.

References

Bates, D. M., & Watts, D. G. (1988). *Nonlinear regression analysis and its applications*. New York, NY: Wiley. http://dx.doi.org/10.1002/9780470316757

Bauer, D. J. (2003). Estimating multilevel linear models as structural equation models. *Journal of Educational and Behavioral Statistics, 28,* 135–167. http://dx.doi.org/10.3102/10769986028002135

Beal, S. L., & Sheiner, L. B. (1982). Estimating population kinetics. *Critical Reviews in Biomedical Engineering, 8,* 195–222.

Blozis, S. A. (2004). Structured latent curve models for the study of change in multivariate repeated measures. *Psychological Methods, 9,* 334–353. http://dx.doi.org/10.1037/1082-989X.9.3.334

Blozis, S. A. (2007a). A second-order structured latent curve model for longitudinal data. In K. van Montfort, H. Oud, & A. Satorra (Eds.), *Longitudinal models in the behavioural and related sciences* (pp. 189–214). Mahwah, NJ: Erlbaum.

Blozis, S. A. (2007b). On fitting nonlinear latent curve models to multiple variables measured longitudinally. *Structural Equation Modeling, 14,* 179–201. http://dx.doi.org/10.1080/10705510709336743

Blozis, S. A. (2012). Nonlinear growth models. In B. Laursen, T. D. Little, & N. A. Card (Eds.), *Handbook of developmental research methods* (pp. 445–463). New York, NY: Guilford Press.

Blozis, S. A., Conger, K. J., & Harring, J. R. (2007). Nonlinear latent curve models for multivariate longitudinal data. *International Journal of Behavioral Development, 31,* 340–346. http://dx.doi.org/10.1177/0165025407077755

Blozis, S. A., & Cudeck, R. (1999). Conditionally linear mixed-effects models with latent variable covariates. *Journal of Educational and Behavioral Statistics, 24,* 245–270. http://dx.doi.org/10.2307/1165324

Blozis, S. A., Harring, J. R., & Mels, G. (2008). Using LISREL to fit nonlinear latent curve models. *Structural Equation Modeling, 15,* 346–379. http://dx.doi.org/10.1080/10705510801922639

Bollen, K. A., & Curran, P. J. (2006). *Latent curve models: A structural equation perspective*. Hoboken, NJ: Wiley.

Bovaird, J. A. (2007). Multilevel structural equation models for contextual factors. In T. D. Little, J. A. Bovaird, & N. A. Card (Eds.), *Modeling contextual effects in longitudinal studies* (pp. 149–182). Mahwah, NJ: Erlbaum.

Browne, M. W. (1993). Structured latent curve models. In C. M. Cuadras & C. R. Rao (Eds.), *Multivariate analysis: Future directions 2* (pp. 171–197). Amsterdam, Netherlands: Elsevier-North-Holland. http://dx.doi.org/10.1016/B978-0-444-81531-6.50016-7

Browne, M. W., & du Toit, S. H. C. (1991). Models for learning data. In L. M. Collins & J. L. Horn (Eds.), *Best methods for the analysis of change* (pp. 47–68). Washington, DC: American Psychological Association.

Choi, J., Harring, J. R., & Hancock, G. R. (2009). Latent growth modeling for logistic response functions. *Multivariate Behavioral Research, 44,* 620–645. http://dx.doi.org/10.1080/00273170903187657

Crockett, P., Harvey, E., Guo, X., Francis, D., & Brouwers, P. (2005). *A mixed effects model with splines with random knot locations*. Paper presented at the 133rd Annual Meeting & Exposition of the American Public Health Association, Philadelphia, PA.

Cudeck, R. (1996). Mixed-effects models in the study of individual differences with repeated measures data. *Multivariate Behavioral Research, 31,* 371–403. http://dx.doi.org/10.1207/s15327906mbr3103_6

Cudeck, R., & du Toit, S. H. C. (2002). A version of quadratic regression with interpretable parameters. *Multivariate Behavioral Research, 37,* 501–519. http://dx.doi.org/10.1207/S15327906MBR3704_04

Cudeck, R., & du Toit, S. H. C. (2003). Nonlinear multilevel models for repeated measures data. In N. Duan & S. P. Reise (Eds.), *Multilevel modeling: Methodological advances, issues and applications* (pp. 1–24). Mahwah, NJ: Erlbaum.

Cudeck, R., & Harring, J. R. (2007). Analysis of nonlinear patterns of change with random coefficient models. *Annual Review of Psychology, 58,* 615–637. http://dx.doi.org/10.1146/annurev.psych.58.110405.085520

Cudeck, R., & Klebe, K. J. (2002). Multiphase mixed-effects models for repeated measures data. *Psychological Methods, 7,* 41–63. http://dx.doi.org/10.1037/1082-989X.7.1.41

Curran, P. J. (2003). Have multilevel models been structural equation models all along? *Multivariate Behavioral Research, 38,* 529–569. http://dx.doi.org/10.1207/s15327906mbr3804_5

Davidian, M., & Giltinan, D. M. (1995). *Nonlinear models for repeated measurement data*. Boca Raton, FL: CRC Press.

Davidian, M., & Giltinan, D. M. (2003). Nonlinear models for repeated measurement data: An overview and update. *Journal of Agricultural Biological & Environmental Statistics, 8,* 387–419. http://dx.doi.org/10.1198/1085711032697

Demidenko, E. (2004). *Mixed models: Theory and applications*. Hoboken, NJ: Wiley. http://dx.doi.org/10.1002/0471728438

Dominicus, A., Ripatti, S., Pedersen, N. L., & Palmgren, J. (2008). A random change point model for assessing variability in repeated measures of cognitive function. *Statistics in Medicine, 27,* 5786–5798. http://dx.doi.org/10.1002/sim.3380

Flora, D. B. (2008). Specifying piecewise latent trajectory models for longitudinal data. *Structural Equation Modeling, 15,* 513–533. http://dx.doi.org/10.1080/10705510802154349

Ghisletta, P., Kennedy, K. M., Rodrigue, K. M., Lindenberger, U., & Raz, N. (2010). Adult age differences and the role of cognitive resources in perceptual-motor skill acquisition: Application of a multilevel negative exponential model. *The Journals of Gerontology: Series B: Psychological Sciences and Social Sciences, 65*B, 163–173. http://dx.doi.org/10.1093/geronb/gbp126

Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. *Philosophical Transactions of the Royal Society of London, 115,* 513–583. http://dx.doi.org/10.1098/rstl.1825.0026

Grimm, K. J., & Ram, N. (2009). Non-linear growth models in Mplus and SAS. *Structural Equation Modeling, 16,* 676–701. http://dx.doi.org/10.1080/10705510903206055

Grimm, K. J., Ram, N., & Estabrook, R. (2010). Nonlinear structured growth mixture models in Mplus and OpenMx. *Multivariate Behavioral Research, 45,* 887–909. http://dx.doi.org/10.1080/00273171.2010.531230

Grimm, K. J., Ram, N., & Hamagami, F. (2011). Nonlinear growth curves in developmental research. *Child Development, 82,* 1357–1371. http://dx.doi.org/10.1111/j.1467-8624.2011.01630.x

Grimm, K. J., & Widaman, K. F. (2010). Residual structures in latent growth curve modeling. *Structural Equation Modeling, 17,* 424–442. http://dx.doi.org/10.1080/10705511.2010.489006

Gurka, M. J., Edwards, L. J., & Muller, K. E. (2011). Avoiding bias in mixed model inference for fixed effects. *Statistics in Medicine, 30,* 2696–2707. http://dx.doi.org/10.1002/sim.4293

Hailwood, A. J., & Horrobin, S. (1946). Absorption of water by polymers: Analysis in terms of a simple model. *Transactions of the Faraday Society, 42,* B084–B092. http://dx.doi.org/10.1039/tf946420b084

Harring, J. R. (2009). A nonlinear mixed effects model for latent variables. *Journal of Educational and Behavioral Statistics, 34,* 293–318. http://dx.doi.org/10.3102/1076998609332750

Harring, J. R., & Blozis, S. A. (2014). Fitting correlated residual error structures in nonlinear mixed-effects models using SAS PROC NL-MIXED. *Behavior Research Methods, 46,* 372–384. http://dx.doi.org/10.3758/s13428-013-0397-z

Harring, J. R., Cudeck, R., & du Toit, S. H. C. (2006). Fitting partially nonlinear random coefficient models as SEMs. *Multivariate Behavioral Research, 41,* 579–596. http://dx.doi.org/10.1207/s15327906mbr4104_7

Harring, J. R., Kohli, N., Silverman, R. D., & Speece, D. L. (2012). A second-order conditionally linear mixed effects model with observed and latent variable covariates. *Structural Equation Modeling, 19,* 118–136. http://dx.doi.org/10.1080/10705511.2012.634729

Huet, S., Bouvier, A., Poursat, M. A., & Jolivet, E. (2004). *Statistical tools for nonlinear regression*. New York, NY: Springer-Verlag.

Hunt, R. (1982). *Plant growth curves: The functional approach to plant growth analysis*. London, UK: Edward Arnold.

Jenss, R. M., & Bayley, N. (1937). A mathematical method for studying the growth of a child. *Human Biology, 9,* 556–563.

Kanfer, R., & Ackerman, P. L. (1989). Motivation and cognitive abilities: An integrative/aptitude-treatment interaction approach to skill acquisition. *Journal of Applied Psychology, 74,* 657–690. http://dx.doi.org/10.1037/0021-9010.74.4.657

Karkach, A. S. (2006). Trajectories and models of individual growth. *Demographic Research, 15,* 347–400. http://dx.doi.org/10.4054/DemRes.2006.15.12

Kelley, K. (2009). The average rate of change for continuous time models. *Behavior Research Methods, 41,* 268–278. http://dx.doi.org/10.3758/BRM.41.2.268

Kelley, K., & Maxwell, S. E. (2008). Delineating the average rate of change in longitudinal models. *Journal of Educational and Behavioral Statistics, 33,* 307–332. http://dx.doi.org/10.3102/1076998607306074

Kohli, N. (2011). *Estimating unknown knots in piecewise linear-linear latent growth mixture models* (Unpublished doctoral dissertation). University of Maryland, College Park, MD.

Kohli, N., & Harring, J. R. (2013). Modeling growth in latent variables using a piecewise function. *Multivariate Behavioral Research, 48,* 370–397. http://dx.doi.org/10.1080/00273171.2013.778191

Kohli, N., Harring, J. R., & Hancock, G. R. (2013, April). *A two-phase linear-linear piecewise growth mixture model*. Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA.

Laird, N. M., & Ware, J. H. (1982). Random-effects models for longitudinal data. *Biometrics, 38,* 963–974. http://dx.doi.org/10.2307/2529876

Leech, J. W., & Ferguson, I. S. (1981). Comparison of yield models for unthinned stands of radiata pine. *Australian Forestry Research, 11,* 231–245.

Li, F., Duncan, T. E., Duncan, S. C., & Hops, H. (2001). Piecewise growth mixture modeling of adolescent alcohol use data. *Structural Equation Modeling, 8,* 175–204. http://dx.doi.org/10.1207/S15328007SEM0802_2

Lindstrom, M. L., & Bates, D. M. (1990). Nonlinear mixed effects models for repeated measures data. *Biometrics, 46,* 673–687. http://dx.doi.org/10.2307/2532087

McArdle, J. J., & Nesselroade, J. R. (2003). Growth curve analysis in contemporary psychological research. In J. A. Schinka & W. F. Velicer (Eds.), *Handbook of psychology: Vol. 2. Research methods in psychology* (pp. 447–480). Hoboken, NJ: Wiley. http://dx.doi.org/10.1002/0471264385.wei0218

McArdle, J. J., & Wang, L. (2008). Modeling age-based turning points in longitudinal life-span growth curves of cognition. In P. Cohen (Ed.), *Applied data analytic techniques for turning points research* (pp. 105–127). New York, NY: Routledge.

Mead, R., & Pike, D. J. (1975). A review of response surface methodology from a biometric viewpoint. *Biometrics, 31,* 803–851. http://dx.doi.org/10.2307/2529809

Meredith, W., & Tisak, J. (1990). Latent curve analysis. *Psychometrika, 55,* 107–122. http://dx.doi.org/10.1007/BF02294746

Muniz Terrera, G., van den Hout, A., & Matthews, F. E. (2011). Random change point models: Investigating cognitive decline in the presence of missing data. *Journal of Applied Statistics, 38,* 705–716. http://dx.doi.org/10.1080/02664760903563668

Muthén, B. (2000). Methodological issues in random coefficient growth modeling using a latent variable framework: Applications to the development of heavy drinking. In J. Rose, L. Chassin, C. Presson, & J. Sherman (Eds.), *Multivariate applications in substance use research* (pp. 113–140). Hillsdale, NJ: Erlbaum.

Muthén, B., & Asparouhov, T. (2011). Beyond multilevel regression modeling: Multilevel analysis in a general latent variable framework. In J. Hox & J. K. Roberts (Eds.), *Handbook of advanced multilevel analysis* (pp. 15–40). New York, NY: Taylor and Francis.

Muthén, L. K., & Muthén, B. O. (1998–2014). *Mplus user's guide* (7th ed.). Los Angeles, CA: Author.

Neale, M. C., & McArdle, J. J. (2000). Structured latent growth curves for twin data. *Twin Research, 3,* 165–177. http://dx.doi.org/10.1375/136905200320565454

Nelder, J. A. (1966). Inverse polynomials. *Biometrics, 22,* 128–141. http://dx.doi.org/10.2307/2528220

Obeid, O. A., Dimachkie, S., & Hlais, S. (2010). Increased phosphorus content of preload suppresses ad libitum energy intake at subsequent meal. *International Journal of Obesity, 34,* 1446–1448. http://dx.doi.org/10.1038/ijo.2010.74

Pinheiro, J., & Bates, D. (2000). *Mixed-effects models in S and S-Plus*. New York, NY: Springer. http://dx.doi.org/10.1007/978-1-4419-0318-1

Preacher, K. J., & Hancock, G. R. (2012). On interpretable reparameterizations of linear and nonlinear latent growth curve models. In J. R. Harring & G. R. Hancock (Eds.), *Advances in longitudinal methods in the social and behavioral sciences* (pp. 25–58). Charlotte, NC: Information Age Publishing.

Ratkowsky, D. A. (1983). *Nonlinear regression modeling: A unified practical approach*. New York, NY: Marcel Dekker.

Ratkowsky, D. A. (1990). *Handbook of nonlinear regression models*. New York, NY: Marcel Dekker.

Rausch, J. R. (2004). *Designing longitudinal studies of negative exponential growth according to the reliabilities of growth parameter estimators*. Unpublished thesis, University of Notre Dame, Notre Dame, IN.

Rausch, J. R. (2008). *Parametrizations for the negative exponential growth model*. Oral presentation given at International Meeting of the Psychometric Society, Durham, NH.

Rovine, M. J., & Molenaar, P. C. M. (1998). A nonstandard method for estimating a linear growth model in LISREL. *International Journal of Behavioral Development, 22,* 453–473. http://dx.doi.org/10.1080/016502598384225

Rovine, M. J., & Molenaar, P. C. M. (2000). A structural modeling approach to a multilevel random coefficients model. *Multivariate Behavioral Research, 35,* 51–88. http://dx.doi.org/10.1207/S15327906MBR3501_3

Rovine, M. J., & Molenaar, P. C. M. (2003). Estimating analysis of variance models as structural equation models. In B. H. Pugesek, A. Tomer, & A. von Eye (Eds.), *Structural equation modeling: Applications in ecological and evolutionary biology* (pp. 235–280). Cambridge, UK: Cambridge University Press. http://dx.doi.org/10.1017/CBO9780511542138.011

Seber, G. A. F., & Wild, C. J. (1989). *Nonlinear regression*. New York, NY: Wiley. http://dx.doi.org/10.1002/0471725315

Serroyen, J., Molenberghs, G., Verbeke, G., & Davidian, M. (2009). Non-linear models for longitudinal data. *The American Statistician, 63,* 378–388. http://dx.doi.org/10.1198/tast.2009.07256

Singer, J. D., & Willett, J. B. (2003). *Applied longitudinal data analysis: Modeling change and event occurrence*. New York, NY: Oxford University Press. http://dx.doi.org/10.1093/acprof:oso/9780195152968.001.0001

Sit, V., & Poulin-Costello, M. (1994). *Catalog of curves for curve fitting*. Victoria, BC: Crown.

Sweda, T. (1984). Theoretical growth equations and their applications in forestry. *Bulletin of the Nagoya University Forests, 7,* 149–260.

Tabatabai, M., Williams, D. K., & Bursac, Z. (2005). Hyperbolastic growth models: Theory and application. *Theoretical Biology & Medical Modelling, 2,* 14. http://dx.doi.org/10.1186/1742-4682-2-14

Vandergrift, N. A. (2004). *Latent curve models for nonlinear target functions: A review and simulation* (Unpublished doctoral dissertation). University of North Carolina at Chapel Hill, Chapel Hill, NC.

Vonesh, E. F., & Chinchilli, V. M. (1997). *Linear and nonlinear models for the analysis of repeated measurements*. New York, NY: Marcel Dekker.

Wall, M. M., & Amemiya, Y. (2007). A review of nonlinear factor analysis and nonlinear structural equation modeling. In R. Cudeck & R. C. MacCallum (Eds.), *Factor analysis at 100: Historical developments and future directions* (pp. 337–362). Mahwah, NJ: Erlbaum.

Wang, L., & McArdle, J. J. (2008). A simulation study comparison of Bayesian estimation with conventional methods for estimating unknown change points. *Structural Equation Modeling, 15,* 52–74. http://dx.doi.org/10.1080/10705510701758265

Widaman, K. F., Helm, J. L., Castro-Schilo, L., Pluess, M., Stallings, M. C., & Belsky, J. (2012). Distinguishing ordinal and disordinal interactions. *Psychological Methods, 17,* 615–622. http://dx.doi.org/10.1037/a0030003

Willett, J. B. (1989). Questions and answers in the measurement of change. In E. Z. Rothkopf (Ed.), *Review of research in education* (Vol. 15, pp. 345–422). Washington, DC: American Education Research Association.

Winsor, C. P. (1932). The Gompertz curve as a growth curve. *PNAS Proceedings of the National Academy of Sciences, 18,* 1–8. http://dx.doi.org/10.1073/pnas.18.1.1

Yang, R. C., Kozak, A., & Smith, J. H. G. (1978). The potential of Weibull-type functions as flexible growth curves. *Canadian Journal of Forest Research, 8,* 424–431. http://dx.doi.org/10.1139/x78-062

Zeger, S. L., Liang, K.-Y., & Albert, P. S. (1988). Models for longitudinal data: A generalized estimating equation approach. *Biometrics, 44,* 1049–1060. http://dx.doi.org/10.2307/2531734

Zerbe, G. O. (1979). Randomization analysis of the completely randomized design extended to growth and response curves. *Journal of the American Statistical Association, 74,* 215–221. http://dx.doi.org/10.1080/01621459.1979.10481640

Zhang, Z., McArdle, J. J., & Nesselroade, J. R. (2012). Growth rate models: Emphasizing growth rate analysis through growth curve modeling. *Journal of Applied Statistics, 39,* 1241–1262. http://dx.doi.org/10.1080/02664763.2011.644528

Appendix

Specification of a Reparameterized Quadratic Curve

Consider the common quadratic curve

$$y = \theta_0 + \theta_1 t + \theta_2 t^2. \qquad \text{(A1)}$$

Many authors have noted the difficulty of lending meaning to the parameters of traditional polynomial functions. Cudeck and du Toit (2002) suggest a reparameterization of Equation A1 that contains more easily interpretable parameters. The reparameterized model is

$$y = \alpha_y - (\alpha_y - \alpha_0)\left(\frac{t}{\alpha_t} - 1\right)^2. \qquad \text{(A2)}$$

Here, $\boldsymbol{\alpha}_0$ is the value of y when $t = 0$ (the intercept), $\boldsymbol{\alpha}_t$ is the value of t that lies at the minimum or maximum of the curve, and $\boldsymbol{\alpha}_y$is the value of y at that value of t.

Here, we show how to specify this model in SEM using both the structured latent curve approach and our modification of it. We might hypothetically fit the model in Equation A2 to Kanfer and Ackerman's (1989) air traffic controller learning task data, consisting of nine equally spaced repeated measures. First, the derivatives of y with respect to each parameter, evaluated at the population point, are

$$\begin{aligned} \partial y/\partial \alpha_0 &= (t/\mu_t - 1)^2 \\ \partial y/\partial \alpha_t &= 2t(\mu_y - \mu_0)(t - \mu_t)/\mu_t^3 \\ \partial y/\partial \alpha_y &= 1 - (t/\mu_t - 1)^2 \end{aligned} \qquad \text{(A3)}$$

The linearized model may then be expressed in matrix form as

$$\begin{aligned} \mathbf{y}_j &= \mathbf{f}(\boldsymbol{\theta},\mathbf{t}) + \boldsymbol{\Lambda}\boldsymbol{\eta}_j + \varepsilon_j \\ &= \boldsymbol{\Lambda}\boldsymbol{\alpha} + \boldsymbol{\Lambda}\boldsymbol{\eta}_j + \varepsilon_j \end{aligned}$$

where $\mathbf{f}(\boldsymbol{\theta},\mathbf{t})$ is the target function and the elements of $\boldsymbol{\Lambda}$ are the derivatives in Equation A3, with time values substituted for t:

(*Appendix continues*)

$$\mathbf{\Lambda} = \left[\frac{\partial \mathbf{y}}{\partial \alpha_0} \quad \frac{\partial \mathbf{y}}{\partial \alpha_t} \quad \frac{\partial \mathbf{y}}{\partial \alpha_y} \right] \Bigg|_{\boldsymbol{\mu}}$$

$$= \begin{bmatrix} (0/\mu_t - 1)^2 & 2(0)(\mu_y - \mu_0)(0 - \mu_t)/\mu_t^3 & 1 - (0/\mu_t - 1)^2 \\ (1/\mu_t - 1)^2 & 2(1)(\mu_y - \mu_0)(1 - \mu_t)/\mu_t^3 & 1 - (1/\mu_t - 1)^2 \\ (2/\mu_t - 1)^2 & 2(2)(\mu_y - \mu_0)(2 - \mu_t)/\mu_t^3 & 1 - (2/\mu_t - 1)^2 \\ (3/\mu_t - 1)^2 & 2(3)(\mu_y - \mu_0)(3 - \mu_t)/\mu_t^3 & 1 - (3/\mu_t - 1)^2 \\ (4/\mu_t - 1)^2 & 2(4)(\mu_y - \mu_0)(4 - \mu_t)/\mu_t^3 & 1 - (4/\mu_t - 1)^2 \\ (5/\mu_t - 1)^2 & 2(5)(\mu_y - \mu_0)(5 - \mu_t)/\mu_t^3 & 1 - (5/\mu_t - 1)^2 \\ (6/\mu_t - 1)^2 & 2(6)(\mu_y - \mu_0)(6 - \mu_t)/\mu_t^3 & 1 - (6/\mu_t - 1)^2 \\ (7/\mu_t - 1)^2 & 2(7)(\mu_y - \mu_0)(7 - \mu_t)/\mu_t^3 & 1 - (7/\mu_t - 1)^2 \\ (8/\mu_t - 1)^2 & 2(8)(\mu_y - \mu_0)(8 - \mu_t)/\mu_t^3 & 1 - (8/\mu_t - 1)^2 \end{bmatrix}$$

In the traditional SLCM approach (Browne, 1993), the researcher determines what elements of the factor mean vector $\boldsymbol{\alpha}$ are estimated or fixed to zero by solving the linear equation $E[\mathbf{y}_j] = \mathbf{f}(\boldsymbol{\theta},\mathbf{t}) = \mathbf{\Lambda}\boldsymbol{\alpha}$:

$$\begin{aligned} \mathbf{f}(\boldsymbol{\theta},\mathbf{t}) &= \mathbf{\Lambda}\boldsymbol{\alpha} \\ (\mathbf{\Lambda}'\mathbf{\Lambda})^{-1}\mathbf{\Lambda}'\mathbf{f}(\boldsymbol{\theta},\mathbf{t}) &= (\mathbf{\Lambda}'\mathbf{\Lambda})^{-1}\mathbf{\Lambda}'\mathbf{\Lambda}\boldsymbol{\alpha} \\ \boldsymbol{\alpha} &= (\mathbf{\Lambda}'\mathbf{\Lambda})^{-1}\mathbf{\Lambda}'\mathbf{f}(\boldsymbol{\theta},\mathbf{t}) \end{aligned} \tag{A4}$$

Even though all the elements of $\mathbf{\Lambda}$ and $\mathbf{f}(\boldsymbol{\theta},\mathbf{t})$ are known, Equation A4 often is intractably complex because of the nonlinear expressions in the elements of $\mathbf{\Lambda}$ and $\mathbf{f}(\boldsymbol{\theta},\mathbf{t})$. An easier way to determine what elements of $\boldsymbol{\alpha}$ to fix to zero is to note which parameters in Equation A2 enter the function nonlinearly and fix the corresponding factor means to zero. This can be accomplished by noting which derivatives in Equation A3 contain the parameter with respect to which the derivative was taken. Only $\partial y/\partial\alpha_t$ meets this criterion because it contains μ_t, so the second element of $\boldsymbol{\alpha}$ is fixed to zero and the other two are estimated:

$$\boldsymbol{\alpha} = [\ \mu_0\ 0\ \ \mu_y]'.$$

As a check, note that evaluating the product $\mathbf{\Lambda}\boldsymbol{\alpha}$ yields the target function

$$\begin{aligned} \mathbf{\Lambda}\boldsymbol{\alpha} &= \left[\frac{\partial \mathbf{y}}{\partial \alpha_0} \quad \frac{\partial \mathbf{y}}{\partial \alpha_t} \quad \frac{\partial \mathbf{y}}{\partial \alpha_y} \right] \Bigg|_{\boldsymbol{\mu}} \begin{bmatrix} \mu_0 \\ 0 \\ \mu_y \end{bmatrix} \\ &= \mu_0 \frac{\partial \mathbf{y}}{\partial \alpha_0}\Bigg|_{\boldsymbol{\mu}} + \mu_y \frac{\partial \mathbf{y}}{\partial \alpha_y}\Bigg|_{\boldsymbol{\mu}} \\ &= \mu_0(t/\mu_t - 1)^2 + \mu_y(1 - (t/\mu_t - 1)^2) \\ &= \mu_y - (\mu_y - \mu_0)\left(\frac{t}{\mu_t} - 1\right)^2. \end{aligned}$$

Alternatively, the linearized model may be expressed in matrix form as

$$\begin{aligned} \mathbf{y}_j &= \mathbf{f}(\boldsymbol{\theta},\mathbf{t}) + \mathbf{\Lambda}\boldsymbol{\eta}_j + \varepsilon_j \\ &= \boldsymbol{\tau} + \mathbf{\Lambda}\boldsymbol{\eta}_j + \varepsilon_j, \end{aligned}$$

There is no need to determine what elements of $\boldsymbol{\alpha}$ should be fixed to zero because $\boldsymbol{\alpha}$ is not in the model. Rather, the target function $\mathbf{f}(\boldsymbol{\theta},\mathbf{t})$ is coded directly into the intercept vector $\boldsymbol{\tau}$.

For the quadratic function applied to Kanfer and Ackerman's (1989) data, it does not matter which specification is used; both yield identical results. We include Mplus code in the online supplemental appendix for readers to demonstrate the equivalence of these methods.[19] The modified approach arguably is easier to implement and to understand, and it can be used in situations in which the classic approach cannot (e.g., the Qinv function considered earlier).

[19] See online supplemental Appendix 7 for Mplus syntax, and see online Appendices 14 and 15 for annotated output.

Received May 8, 2013
Revision received October 3, 2014
Accepted October 10, 2014 ■