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Multiple Local Solutions and Geomin Rotation

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ABSTRACT

In exploratory factor analysis, factor rotation is conducted to improve model interpretability. A promising and increasingly popular factor rotation method is geomin rotation. Geomin rotation, however, frequently encounters multiple local solutions. We report a simulation study that explores the frequency of local solutions in geomin rotation and the implications of such phenomena. The findings include: (1) multiple local solutions exist for geomin rotation in a variety of situations; (2) $\varepsilon = .01$ provides satisfactory rotated factor loadings in most situations; (3) 100 random starts appear sufficient to examine the multiple solution phenomenon; and (4) a population global solution may correspond to a sample local solution rather than the sample global solution.

KEYWORDS

Factor analysis; factor rotation; geomin; local solutions

Introduction

Exploratory factor analysis (EFA) is one of the most popular multivariate statistical procedures in the social and behavioral sciences. The primary goal of EFA is to explain correlations among a number of manifest variables using a much smaller number of common factors. Relations between manifest variables and common factors are shown by factor loading matrices. Because infinitely many factor loading matrices produce identical fit to a particular data set, factor rotation is employed to select just one factor loading matrix for interpretation. Factor rotation (Browne, 2001) is often conducted by optimizing a scalar function of the factor loading matrix. The scalar function is referred to as a factor rotation criterion function, the application of which is intended to enhance the interpretability of the factor loading matrix. Popular choices of factor rotation criteria are varimax (Kaiser, 1958), oblimin (Jennrich & Sampson, 1966), and the Crawford-Ferguson family (Crawford & Ferguson, 1970).[1]

The factor rotation criterion *geomin* (Asparouhov & Muthén, 2009; Browne, 2001; Yates, 1987) has recently received substantial attention. Browne (2001) demonstrated geomin rotation using the Thurstone box data (Thurstone, 1947), a well-known difficult problem for which many rotation methods fail. A geomin rotated factor loading matrix matches the three dimensions of the data set. Geomin rotation also produced satisfactory results in several empirical data sets (Browne, 2001, Table 8; Schmitt & Sass, 2011, Table 5). Parameter estimates obtained with geomin rotation are reported to be comparable with confirmatory factor analysis (Asparouhov & Muthén, 2009) and to be unbiased (Sass & Schmitt, 2010, Table 3) in simulation studies in which the factor loading matrices adhere to simple structure. Thus, geomin rotation is recommended for its ability to produce factor loadings and factor correlations similar to those of confirmatory factor analysis without having to specify the factor loading pattern (Schmitt, 2011). The use of geomin rotation is facilitated by its availability in computer software, such as M*plus* (Muthén & Muthén, 1998-2017), CEFA (Browne, Cudeck, Tateneni, & Mels, 2010), and different implementations (MATLAB, R, SAS PROC IML, and SPSS) of the gradient projection algorithm (Bernaards & Jennrich, 2005). In particular, geomin rotation is the default rotation criterion for EFA in M*plus*.

An often unnoticed phenomenon in factor rotation is the existence of local solutions. Optimization of a factor rotation criterion requires an iterative procedure, which improves the factor rotation criterion function value from a starting value in a step-by-step manner. Ideally, the iterative procedure always leads to the optimal solution regardless of starting values. A local solution occurs if a convergent solution differs from the optimal solution. Although the issue of local solutions was noted for varimax and quartimax 50 years ago (Gebhardt, 1968), varimax and quartimax local solutions have not been reported for empirical studies. However, factor analysts frequently

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[1]Promax (Hendrickson & White, 1964) is another popular choice, but it is a two-stage method. Varimax rotation is carried out first to produce a factor reference matrix. A procrustean rotation is then carried out to match the reference matrix.

encounter multiple local solutions with geomin rotation (Asparouhov & Muthén, 2009; Browne, 2001).

We have two goals. First, we examine factors affecting the occurrence of local solutions with geomin rotation. Second, we assess implications of the possible existence of local solutions on the interpretation of EFA results. In particular, it is not guaranteed that the sample global solution corresponds to the population global solution. Therefore, a factor analyst must examine local solutions in addition to the global solution.

The rest of the article is organized as follows. We first briefly describe EFA models and their estimation. We pay close attention to factor rotation and its implementation. We then examine the frequency and nature of local solutions with both empirical data and simulated data. In particular, we explore how features of the EFA model and choices of a geomin rotation parameter affect occurrences of local solutions. We conclude with several remarks and recommendations.

EFA models and their estimation

The factor analysis model specifies that the manifest variables are weighted sums of common factors and unique factors

$$\boldsymbol{y} = \boldsymbol{\Lambda} \boldsymbol{f} + \boldsymbol{u} \tag{1}$$

Here, $\boldsymbol{y}$ is a $p \times 1$ vector of standardized manifest variables for a typical individual, $\boldsymbol{f}$ is a $m \times 1$ vector of common factors, $\boldsymbol{\Lambda}$ is a $p \times m$ matrix of factor loadings relating p manifest variables to m common factors, and $\boldsymbol{u}$ is a $p \times 1$ vector of unique factors. The p unique factors $\boldsymbol{u}$ are assumed to be uncorrelated with the common factors $\boldsymbol{f}$ and to be uncorrelated among themselves.

Estimation of EFA consists of two steps. The first step is to estimate unrotated factor loadings $\boldsymbol{A}$. Popular methods of obtaining unrotated factor loadings $\boldsymbol{A}$ are maximum likelihood (ML) and ordinary least squares (OLS) estimation. The EFA model is not identified: infinitely many sets of parameter estimates produce the same fit to data. Additional constraints are added to identify the EFA model. In OLS estimation, $\boldsymbol{A}'\boldsymbol{A}$ is constrained to be a diagonal matrix. In ML estimation, $\boldsymbol{A}'\boldsymbol{\Psi}^{-1}\boldsymbol{A}$ is constrained to be a diagonal matrix. Here, the $p \times p$ diagonal matrix $\boldsymbol{\Psi}$ contains the variances of unique factors $\boldsymbol{u}$. Because these constraints are imposed for mathematical convenience, the unrotated factor loading matrix $\boldsymbol{A}$ is seldom interpretable.

The second step of EFA estimation is to rotate $\boldsymbol{A}$ to improve interpretability. A rotated factor loading matrix $\boldsymbol{\Lambda}$ and a factor correlation matrix $\boldsymbol{\Phi}$ are computed from the unrotated factor loading matrix $\boldsymbol{A}$ and an $m \times m$ rotation matrix $\boldsymbol{T}$

$$\boldsymbol{\Lambda} = \boldsymbol{A}\boldsymbol{T} \quad \text{and} \quad \boldsymbol{\Phi} = \boldsymbol{T}^{-1}\boldsymbol{T}'^{-1}. \tag{2}$$

The rotation matrix $\boldsymbol{T}$ is chosen to optimize a rotation criterion function $Q(\boldsymbol{\Lambda})$. The value of the rotation criterion function is intended to reflect the interpretability of the factor loading matrix. Two types of factor rotation are available: factors are allowed to be correlated in oblique rotation; factors are not allowed to be correlated in orthogonal rotation.

Factor rotation can also be considered as imposing constraints to identify EFA models. Let $\frac{dQ}{d\boldsymbol{\Lambda}}$ denote a $p \times m$ matrix that contains the derivatives of the rotation criterion function $Q(\boldsymbol{\Lambda})$ with regard to rotated factor loadings. In orthogonal rotation, $\boldsymbol{\Lambda}'\frac{dQ}{d\boldsymbol{\Lambda}}$ is constrained to be symmetric (Archer & Jennrich, 1973); in oblique rotation, $\boldsymbol{\Lambda}'\frac{dQ}{d\boldsymbol{\Lambda}}\boldsymbol{\Phi}^{-1}$ is constrained to be diagonal (Jennrich, 1973). For simple presentation, let $\boldsymbol{\theta}$ denote a $q \times 1$ vector that contains pm rotated factor loadings, $m(m-1)/2$ factor correlations, and p unique factor variances. Note that the effective number of parameters in an EFA model is less than q because constraints are imposed for identification purposes.

Geomin rotation (Asparouhov & Muthén, 2009; Browne, 2001; Yates, 1987) has received substantial attention

$$\mathbf{Q}(\boldsymbol{\Lambda}) = \sum_{i=1}^{p}\left[\prod_{j=1}^{m}\left(\lambda_{ij}^2 + \varepsilon\right)\right]^{\frac{1}{m}} \tag{3}$$

where λ_{ij} is a factor loading and ε is a small positive quantity. It is approximately the sum of geometric means of squared factor loadings for each manifest variable. According to Thurstone's simple structure, at least one of the m factor loadings in each row needs to be zero. Without ε, the geometric mean of the m squared factor loadings would be zero if one of the m loadings is exactly zero. This would create an indeterminacy problem because other nonzero loadings would no longer contribute to the geometric mean. The small quantity ε is added to reduce indeterminacy. Therefore, a small loading results in a lower value of the geometric mean of the squared factor loadings in the respective row. Note that the geomin rotation criterion is a complexity function: a lower value corresponds to a more interpretable factor loading matrix.

Minimizing the geomin rotation criterion with regard to the rotation matrix $\boldsymbol{T}$ is not a simple task because it is a nonlinear function of multiple variables. The minimization process is an iterative one. It starts with some initial values and improves the geomin rotation criterion value in a step-by-step manner. The iterative procedure stops when the geomin rotation criterion cannot be made smaller. The rotated factor loading matrix $\boldsymbol{\Lambda}$ and the factor correlation matrix $\boldsymbol{\Phi}$ are then computed using the unrotated factor loading matrix $\boldsymbol{A}$ and the rotation matrix $\boldsymbol{T}$. However, different starting values may result in different sets of $\boldsymbol{\Lambda}$ and $\boldsymbol{\Phi}$. A solution is called the global solution if its corresponding geomin criterion value is the lowest in the whole parameter space; a solution is called a local solution if its corresponding geomin criterion value is the lowest in its neighborhood but is larger than the global solution. Because geomin rotation frequently encounters the issue of multiple local solutions, the factor analyst should carry out factor rotation multiple times from different starting values. Such starting values can be generated by post-multiplying the unrotated factor loading matrix $\boldsymbol{A}$ by random orthogonal matrices.

When different factor loading matrices are compared, they need to be properly aligned to ensure that a given column of $\boldsymbol{\Lambda}$ always refers to the same factor across solutions. Note that column interchange and column reflection are allowed for rotated factor rotation matrices without changing the factor rotation criterion value. Column interchange is the operation that reorders columns of the factor loading matrix; column reflection is the operation that reverses the sign of each and every element in a column of the factor loading matrix. This is referred to as the alignment problem in the factor analysis literature (Jennrich, 2007). We align two factor loading matrices by minimizing the sum of squared differences between the corresponding elements in the two matrices.

Occurrences of local solutions with empirical data sets

The goals of this section are (1) to investigate how different levels of the geomin rotation parameter ε affect occurrences of local solutions and (2) to determine the appropriate number of random starts. The investigation involves two three-factor EFA models and a four-factor EFA model. For each model, we consider four levels of ε (.02, .01, .001, and .0001) and three numbers of random starts ($K = 30$, $K = 100$, and $K = 1{,}000$). When multiple solutions exist, we label the rotated factor loading matrices using $\boldsymbol{\Lambda}^{\mathrm{I}}$, $\boldsymbol{\Lambda}^{\mathrm{II}}$, $\boldsymbol{\Lambda}^{\mathrm{III}}$, etc., according to their respective geomin rotation function values. Although $\boldsymbol{\Lambda}^{\mathrm{I}}$ has the lowest geomin function value among all solutions, we can never be certain that it is the global solution even with a large number of random starts. The solution $\boldsymbol{\Lambda}^{\mathrm{I}}$ is referred to as "the global solution" for simple presentation, but it is actually just the putative global solution.

Three EFA models

Model M_{3a} is a three-factor model. It is fit to Holzinger's unpublished data set, which is a classic data set in the factor analysis literature. It involves nine variables with $N = 696$ participants (Harman, 1978). The nine variables are "word meaning," "sentence completion," "odd words," "mixed arithmetic," "remainders," "missing numbers," "gloves," "boots," and "hatchets." ML estimation of Model M_{3a} produced a 90% confidence interval (.00, .02) for the RMSEA, which indicates excellent fit.

Model M_{3b} is also a three-factor model. It is fit to Thurstone's box data (Thurstone, 1947), which is another classic data set in the factor analysis literature. The data set was created to demonstrate the robustness of EFA for nonlinear relations. Thurstone measured the height (h), width (w), and length (l) of 30 boxes. He then computed 26 functions from these three dimensions. Some examples of the 26 functions are hw^2, h/l, and $\sqrt{h + w + l}$. Cureton and Mulaik (1975, Table 4) reported an unrotated factor loading matrix for the model. Although both M_{3a} and M_{3b} are three-factor models, the rotated factor loading matrix of M_{3a} is simpler than that of M_{3b}. In M_{3a}, different sets of manifest variables load on each factor, and each manifest variable has only one large loading on a factor. This factor loading pattern is referred to as an independent cluster solution. In contrast, an independent cluster solution does not exist for M_{3b}.

Model M_4 is a four-factor model. It is fit to all 24 variables reported by Holzinger and Swineford (1939). The 24 variables are "visual perception," "cubes," "paper form board," "flags," "general information," "paragraph comprehension," "sentence comprehension," "word classification," "word meaning," "addition," "code," "counting dots," "straight-curved capitals," "word recognition," "number recognition," "figure recognition," "object-number," "number-figure," "figure-word," "deduction," "numerical puzzles," "problem reasoning," "series completion," and "arithmetic problems." ML estimation of Model M_4 produced a 90% confidence interval (.00, .05) for the RMSEA, which supports close fit. The rotated factor loading matrix deviates from an independent cluster solution but is still consistent with simple structure.

Four levels of ε and three levels of random starts

The four levels of geomin rotation parameter ε values are .02, .01, .001, and .0001. Note that the small positive quantity ε is added to alleviate indeterminacy of the geomin rotation criterion caused by exactly zero loadings. Therefore ε needs to be small but not too small. If it is too small, indeterminacy caused by zero loadings cannot be effectively alleviated; if it is too large, it will

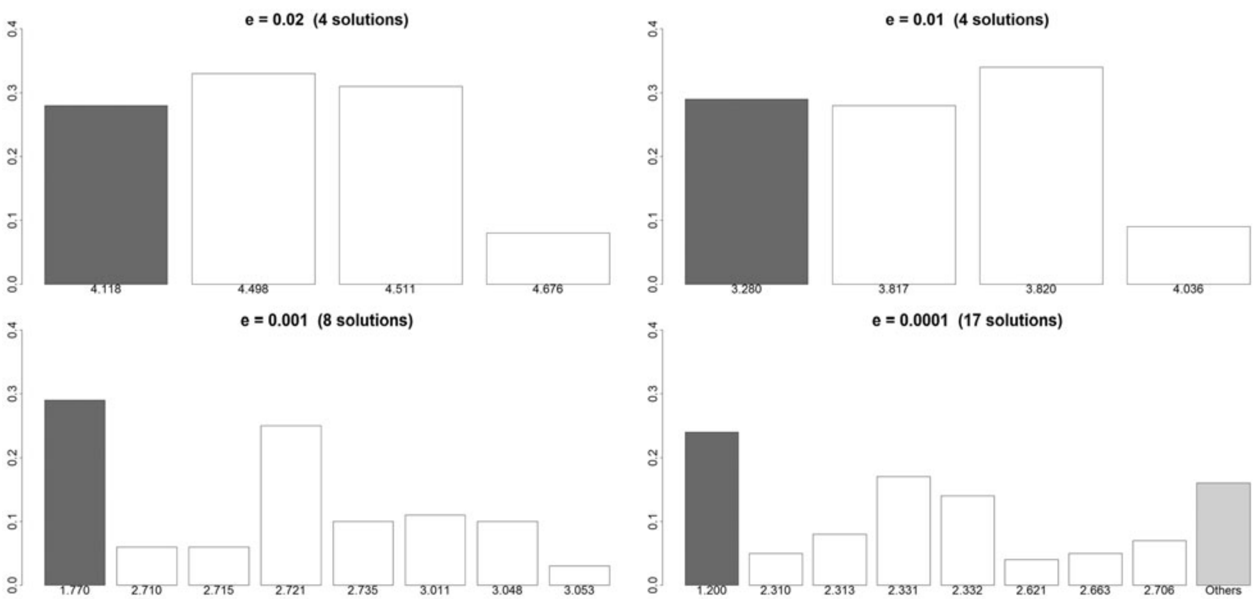

Figure 1. Proportions of sample solutions for three models with 100 random starts at different levels of ε and numbers of random starts.

overwhelm nonzero factor loadings. These four levels of ε are equivalent to increasing the magnitude of a zero factor loading by .14, .10, .03, and .01, respectively. Browne (2001) recommended $\varepsilon = .01$ for three or four factors and higher values for more factors. The default settings in M*plus* (Muthén & Muthén, 1998-2017) are $\varepsilon = .0001$ for two factors, $\varepsilon = .001$ for three factors, and $\varepsilon = .01$ for four or more factors.

Rotated parameters obtained with different levels of ε tend to be different, but such differences are conceptually distinct from statistical bias. When the expected values of parameter estimates are different from the population value, the parameter estimates are biased. In EFA, different rotated factor loading matrices and factor correlations obtained with different levels of ε are mathematically equivalent; they fit data equally well. To evaluate whether such parameter estimates are biased, we compare their expected values with the corresponding population parameters. The ε value should be the same in both the samples and the population. Because sample correlations are consistent estimates of population correlations, geomin rotated parameter estimates are asymptotically unbiased according to Theorem 1 of Yuan and Jennrich (1998).

To study local solutions in geomin rotation, we need to compare the results of factor rotation from multiple starting values. These starting values were randomly generated. Three levels of random starts are considered: $K = 30$, $K = 100$, and $K = 1{,}000$. The default setting in M*plus* (Muthén & Muthén, 1998-2017) is to use $K = 30$ random starts. Browne (2001) used 100 random starts to investigate local solutions. We include the level of $k = 1{,}000$ for comparison.

Results of geomin rotation with empirical data sets

Factor extraction and factor rotation were carried out using CEFA (Browne et al., 2010). Unrotated factor loading matrices were estimated using ML for Holzinger's unpublished data set and 24-test data set. The unrotated factor loading matrix for the Box data was given by Cureton and Mulaik (1975, Table 4).[2]

The number of local solutions at different levels of ε

Because ε is added to alleviate indeterminacy caused by exactly zero loadings, we expect that smaller values of ε deal with indeterminacy less effectively. Figure 1 displays the global solutions and local solutions with $K = 100$ random starts at the four levels of ε for Thurstone's box data. The numbers of solutions are 4, 4, 8, and 17 at $\varepsilon = .02$, $\varepsilon = .01$, $\varepsilon = .001$, and $\varepsilon = .0001$, respectively. Also displayed in Figure 1 are the proportions of these solutions being produced out of the 100 random starts. The global solutions are not necessarily the most frequently occurring solution. Local solutions occur more frequently than the global solutions at $\varepsilon = .02$ and $\varepsilon = .01$. Holzinger's unpublished data set produced only global solutions at $\varepsilon = .02$, $\varepsilon = .01$, and $\varepsilon = .001$; it produced 5 local solutions at $\varepsilon = .0001$. The 24-test data set produced only global solutions at $\varepsilon = .02$ and $\varepsilon = .01$. It produced 12 local solutions at $\varepsilon = .001$ and 83 local solutions at $\varepsilon = .0001$. Figures displaying multiple solutions for Holzinger's unpublished data set and the

[2] The three data sets are also available in R packages: Holzinger's unpublished data set is included in the R package psych (Revelle, 2016, `Harman.Holzinger`); the 24-test data set is included in the R package MBESS (Kelley, 2017, `HS.data`); the box data set is included in the R package GPArotation (Bernaards & Jennrich, 2015, `box26`).

24-test data set are included in the online appendices (https://www3.nd.edu/gzhang3/Papers/LocalGeomin/Geomin.html). The overall conclusion is that the number of local solutions increases as ε decreases. In particular, $\varepsilon = .0001$ seems too small for Thurstone's box data and the 24-test data set.

Comparisons between local solutions and the global solution

When local solutions exist, it is informative to compare them with the global solutions. We demonstrate such comparisons with Thurstone's box data at $\varepsilon = .01$. Four solutions emerged from 100 random starts: the global solution $\hat{\mathbf{\Lambda}}^{\mathrm{I}}$ (with the geomin rotation criterion value 3.280) and three local solutions $\hat{\mathbf{\Lambda}}^{\mathrm{II}}$, $\hat{\mathbf{\Lambda}}^{\mathrm{III}}$, and $\hat{\mathbf{\Lambda}}^{\mathrm{IV}}$ (with the geomin rotation criterion values 3.817, 3.820, and 4.036).

Table 1 presents the global solution ($\hat{\mathbf{\Lambda}}^{\mathrm{I}}$) and a local solution ($\hat{\mathbf{\Lambda}}^{\mathrm{II}}$) side by side. Also presented in Table 1 are the factor loadings and factor correlations obtained from CF-varimax rotation. The global solution agrees with how the manifest variables are created, but the local solution and CF-varimax solution do not. CF-varimax rotation produced only one solution with $K = 100$ random starts, but this solution is not interpretable.

We computed the congruence coefficients (Burt, 1948; Lorenzo-Seva & ten Berge, 2006) between the global solution and the local solution to quantify the relations between corresponding columns in the two matrices. The congruence coefficient is defined as

$$c_j = \frac{\sum_{i=1}^{p} \lambda_{ij,g}\lambda_{ij,l}}{\sqrt{\sum_{i=1}^{p} \lambda_{ij,g}^2}\sqrt{\sum_{i=1}^{p} \lambda_{ij,l}^2}} \tag{4}$$

Here, $\lambda_{ij,g}$ is a factor loading of the global solution, $\lambda_{ij,l}$ is a factor loading of the local solution, $i = 1, 2, \ldots, p$, and $j = 1, 2, \ldots, m$. The higher the value of the congruence coefficient is, the more similar these two factors are. If two factors are perfectly matched, the congruence coefficient is 1.00. MacCallum, Widaman, Zhang, and Hong (1999) interpreted the congruence coefficient in the following ways: .98 to 1.00 = excellent, .92 to .98 = good, .82 to .92 = borderline, .68 to .82 = poor, and below .68 = terrible.

Table 1. The global solution and a local solution at $\varepsilon = .01$ and the CF-varimax solution for Model M_{3b}.

	Factor loadings								
	$\hat{\mathbf{\Lambda}}^{\mathrm{I}}$ (3.280)			$\hat{\mathbf{\Lambda}}^{\mathrm{II}}$ (3.817)			CF-varimax		
	h	l	w	h	l	w	h	l	w
h	.993	−.016	−.010	.654	.660	−.308	.683	.619	−.390
l	.060	.942	.049	−.340	.581	.680	−.199	.725	.552
w	−.001	.060	.965	.582	−.356	.759	.631	−.133	.692
hl	.638	.641	−.013	.132	.815	.254	.245	.881	.119
h^2l	.835	.382	.007	.387	.784	.030	.470	.810	−.090
hl^2	.386	.812	.034	−.079	.728	.483	.053	.838	.345
$2h+2l$	.549	.710	−.020	.039	.798	.323	.160	.876	.188
h^2+l^2	.538	.700	−.007	.044	.780	.329	.163	.859	.194
hw	.596	.001	.646	.800	.138	.306	.846	.260	.215
h^2w	.788	−.021	.418	.792	.346	.066	.832	.407	−.025
hw^2	.444	.026	.858	.823	−.034	.526	.876	.144	.432
$2h+2w$	.557	−.025	.689	.812	.079	.332	.855	.208	.245
h^2+w^2	.525	−.011	.681	.780	.070	.345	.824	.200	.260
lw	−.024	.612	.645	.124	.087	.916	.239	.320	.800
l^2w	−.018	.765	.453	−.060	.260	.879	.070	.472	.755
lw^2	−.028	.440	.784	.284	−.074	.900	.380	.165	.798
$2l+2w$	−.006	.617	.632	.126	.108	.904	.242	.337	.787
l^2+w^2	.016	.617	.600	.120	.135	.874	.236	.357	.757
h/l	.746	−.829	.008	.858	.003	−.794	.762	−.143	−.753
l/h	−.746	.829	−.008	−.858	−.003	.794	−.762	.143	.753
h/w	.817	.007	−.825	.013	.885	−.847	.006	.672	−.866
w/h	−.817	−.007	.825	−.013	−.885	.847	−.006	−.672	.866
l/w	−.008	.849	−.798	−.879	.823	.006	−.789	.769	−.052
w/l	.008	−.849	.798	.879	−.823	−.006	.789	−.769	.052
hwl	.453	.477	.467	.384	.399	.549	.488	.554	.421
$h^2+w^2+l^2$	.341	.527	.487	.302	.345	.632	.410	.516	.506
	Factor correlations								
h	1			1			1		
l	.206	1		.315	1		.180	1	
w	.279	.268	1	.277	.353	1	.157	.197	1

Note. Geomin criterion values in parenthese.

The congruence coefficients for the three factors between the global solution $\hat{\boldsymbol{\Lambda}}^{\mathrm{I}}$ and the local solution $\hat{\boldsymbol{\Lambda}}^{\mathrm{II}}$ are .66, .66, and .74, respectively. Therefore, the global solution $\hat{\boldsymbol{\Lambda}}^{\mathrm{I}}$ and the local solution $\hat{\boldsymbol{\Lambda}}^{\mathrm{II}}$ are very different from each other. The comparisons of the global solution with the other two local solutions $\hat{\boldsymbol{\Lambda}}^{\mathrm{III}}$ and $\hat{\boldsymbol{\Lambda}}^{\mathrm{IV}}$ showed that the other two local solutions are also different from the global solution $\hat{\boldsymbol{\Lambda}}^{\mathrm{I}}$. Note that comparisons between global solution and local solutions are possible only when multiple random starts are used. Without multiple random starts, a local solution is likely to be wrongly regarded as "the" solution of factor rotation.

Additionally, the congruence coefficients for the three factors between the global solution $\hat{\boldsymbol{\Lambda}}^{\mathrm{I}}$ and the CF-varimax solution are .64, .67, and .65, respectively. The global solution $\hat{\boldsymbol{\Lambda}}^{\mathrm{I}}$ and the CF-varimax solution are very different from each other as well.

Model M_{3a} does not produce local solutions at $\varepsilon = .02$, $\varepsilon = .01$, and $\varepsilon = .001$. Model M_4 does have local solutions at $\varepsilon = .001$. The corresponding results for the other two local solutions for Model M_{3b} and all 6 solutions for Model M_4 at $\varepsilon = .001$ with 100 random starts are given in the online appendices.

Global solutions at different levels of ε

When determining the parameter ε for geomin rotation, it is important to examine how changing ε affects the global solution. Table 2 displays the three rotated factor loading matrices of the global solutions at $\varepsilon = .02$, .01, and .001 for Model M_{3b}. The three factor loading matrices are similar. For example, the three factor loadings of the manifest variable h are .992, −.015, and −.006 at $\varepsilon = .02$; they are .993, −.016, and −.010 at $\varepsilon = .01$; and they are .993, −.014, and −.014 under $\varepsilon = .001$. To quantify the similarity of the factor loading matrices, we computed the congruence coefficients for corresponding columns in three rotated factor loading matrices. All nine congruence coefficients are larger than .99. The comparisons of the global solutions at the three levels of ε have a similar pattern in the other three models and the corresponding factor loading matrices, and congruence coefficients are included in the online appendices.

The number of random starts

The use of multiple random starts is essential to detect the existence of local solutions. The results suggest that 30 random starts are sufficient for uncovering the global solutions. However, if the goal is to uncover all possible local solutions, more random starts should be considered. For example, when $\varepsilon = .001$ was used for Model M_4, 30 random starts produced 6 solutions, 100 random starts produced 9 solutions, and 1,000 random starts produced 31 solutions. Therefore we consider $K = 100$ random starts as a compromise between these two goals.

Simulation studies

We further investigate how the geomin rotation parameter ε affects the number and nature of local solutions with

Table 2. Population global solutions at different values of ε for Model M_{3b}.

	$\varepsilon = .02$			$\varepsilon = .01$			$\varepsilon = .001$		
	h	l	w	h	l	w	h	l	w
h	.992	−.015	−.006	.993	−.016	−.010	.993	−.014	−.012
l	.056	.942	.050	.060	.942	.049	.070	.939	.053
w	−.002	.059	.966	−.001	.060	.965	.003	.065	.963
hl	.634	.642	−.010	.638	.641	−.013	.645	.640	−.012
h^2l	.833	.383	.011	.835	.382	.007	.839	.382	.007
hl^2	.381	.813	.036	.386	.812	.034	.394	.810	.037
$2h+2l$	.545	.710	−.017	.549	.710	−.020	.556	.708	−.018
h^2+l^2	.534	.700	−.005	.538	.700	−.007	.545	.698	−.005
hw	.595	.001	.648	.596	.001	.646	.599	.005	.643
h^2w	.787	−.021	.421	.788	−.021	.418	.790	−.017	.415
hw^2	.443	.026	.860	.444	.026	.858	.448	.032	.855
$2h+2w$	.555	−.025	.691	.557	−.025	.689	.559	−.020	.686
h^2+w^2	.524	−.011	.684	.525	−.011	.681	.528	−.006	.679
lw	−.028	.612	.645	−.024	.612	.645	−.016	.614	.646
l^2w	−.022	.766	.453	−.018	.765	.453	−.009	.765	.455
lw^2	−.031	.440	.785	−.028	.440	.784	−.021	.443	.785
$2l+2w$	−.009	.617	.632	−.006	.617	.632	.003	.619	.633
l^2+w^2	.013	.617	.600	.016	.617	.600	.025	.618	.601
h/l	.749	−.830	.011	.746	−.829	.008	.738	−.825	.003
l/h	−.749	.830	−.011	−.746	.829	−.008	−.738	.825	−.003
h/w	.817	.008	−.823	.817	.007	−.825	.814	.003	−.825
w/h	−.817	−.008	.823	−.817	−.007	.825	−.814	−.003	.825
l/w	−.011	.850	−.798	−.008	.849	−.798	−.002	.842	−.792
w/l	.011	−.850	.798	.008	−.849	.798	.002	−.842	.792
hwl	.450	.478	.469	.453	.477	.467	.460	.479	.467
$h^2+w^2+l^2$	.338	.527	.489	.341	.527	.487	.348	.528	.488

simulated data. We also compare sample global solutions with the corresponding population global solutions. In addition, sample size and manifest variable communality are varied in the simulation studies.

The design of simulation studies

Population correlation matrices

We include three EFA models in the simulation studies. These three models are models M_{3a}, M_{3b}, and M_4, which were described in the previous section. For each of the three models, four population correlation matrices were generated according to the four levels of manifest variable communality: low, medium, high, and wide. The model-implied correlation matrices obtained from the original empirical studies were rescaled to satisfy the communality conditions. Communalities are uniformly distributed from .2 to .4 in the low communality condition, from .4 to .6 in the medium communality condition, from .6 to .8 in the high communality condition, and from .2 to .8 in the wide communality condition.

Generation and analysis of simulated samples

The simulation studies involve five levels of sample size: 60, 100, 200, 300, and 500. The levels of communality and sample size resemble their typical values in applied EFA studies (Fabrigar, Wegener, MacCallum, & Strahan, 1999; MacCallum et al., 1999). Combining four EFA models, four levels of communalities, and five levels of sample size produces 80 conditions. For each of the 80 conditions, 1,000 samples were simulated from the corresponding population correlation matrix. Manifest variables are normally distributed in the simulated samples. Unrotated factor loading matrices $\hat{\boldsymbol{A}}$ were extracted using ML estimation from these simulated sample correlation matrices, and they were rotated using geomin rotation with three levels of $\varepsilon = .02$, .01, and .001. One hundred random starts were used in each rotation.

Results of the simulation studies

Different levels of ε and numbers of local solutions

Table 3 reports the median of local solution numbers at the three levels of ε for Model M_{3b}. The median was computed from 1000 simulated samples for each of the 20 conditions. Three observations can be made from Table 3. First, the number of local solutions increases as ε decreases. In particular, $\varepsilon = .001$ produced far more local solutions than $\varepsilon = .02$ and $\varepsilon = .01$. Second, communalities and sample sizes have little influence on the number and frequencies of local solutions. The proportions of simulation samples in which the local solution number is equal to the median number are much larger at $\varepsilon = .02$ and at $\varepsilon = .01$ than those at $\varepsilon =.001$. The influences of different levels of ε, sample size, and communality on the number of local solutions are similar for models M_{3a} and M_4. Tables reporting local solution numbers for models M_{3a}, and M_4 are presented in the online appendices.

Table 3. Median numbers of sample solutions for Model M_{3b}. Proportions of samples with the median number of solutions in parentheses.

Conditions				
N	*h*	$\varepsilon = .02$	$\varepsilon = .01$	$\varepsilon = .001$
60	Low	2 (35.4)	2 (35.1)	7 (15.9)
	Medium	3 (60.6)	3 (51.9)	7 (18.0)
	High	3 (54.9)	4 (33.5)	7 (22.2)
	Wide	2 (37.0)	3 (37.5)	6 (20.0)
100	Low	2 (38.3)	3 (44.9)	6 (20.0)
	Medium	3 (66.5)	3 (55.7)	6 (21.8)
	High	3 (50.0)	4 (39.1)	7 (23.9)
	Wide	2 (42.6)	3 (45.6)	6 (21.7)
200	Low	3 (57.8)	3 (56.2)	6 (23.0)
	Medium	3 (70.0)	3 (51.1)	6 (25.2)
	High	4 (45.2)	4 (40.8)	7 (26.5)
	Wide	2 (48.7)	3 (56.9)	6 (26.5)
300	Low	3 (67.6)	3 (57.0)	5 (28.6)
	Medium	3 (69.2)	3 (50.3)	6 (26.0)
	High	4 (51.6)	4 (42.0)	7 (26.9)
	Wide	3 (47.0)	3 (57.2)	6 (27.5)
500	Low	3 (68.7)	3 (53.2)	6 (25.6)
	Medium	3 (70.4)	4 (42.0)	6 (29.1)
	High	4 (64.3)	4 (50.7)	7 (29.8)
	Wide	2 (51.7)	3 (59.5)	6 (27.7)

Note. N = sample size. *h* = communality.

Global solutions at different levels of ε

We now examine whether different levels of ε substantially change the global solutions in simulated samples. To quantify the differences among the global solutions at $\varepsilon = .02$, .01, and .001, we compute the congruence coefficients among the three rotated factor loading matrices in each simulated sample. We then summarize the similarity of global solutions at different levels of ε of 1,000 simulated samples in each of the 80 conditions by computing the percentage of simulated samples in which the congruence coefficients for all factors are larger than .98. Table 4 reports such percentages for all 80 conditions.

Three observations can be made from Table 4. First, the global solutions at $\varepsilon = .02$ and .01 are more similar to each other than the global solutions at $\varepsilon = .01$ and .001 in models M_{3a} and M_4. For example, when increasing $\varepsilon = .01$ to .02, the percentages of simulated samples in which the congruence coefficients are larger than .98 range from 96.2% to 99.0% for Model M_{3a} even at the sample size of 60; when decreasing $\varepsilon =$ .01 to .001, the corresponding percentages range from 58.3% to 80.0%. The comparisons between the similarity of global solutions at $\varepsilon = .02$ and .01 and the similarity of global solutions at $\varepsilon = .01$ and .001 are less clear in Model M_{3b}. Increasing $\varepsilon = .01$ to .02 tends

Table 4. Percentages of samples with congruence coefficients > .98 across sample global solutions at different values of ε.

Conditions		Pair of ε values					
		M_{3a}		M_{3b}		M_4	
N	h	.02/.01	.01/.001	.02/.01	.01/.001	.02/.01	.01/.001
60	Low	96.6	57.7	88.6	72.6	77.7	34.7
	Medium	96.2	60.2	87.1	75.6	88.7	39.1
	High	99.0	80.0	88.7	88.0	95.1	46.9
	Wide	96.2	60.3	86.9	63.1	85.0	38.0
100	Low	96.4	66.2	86.3	73.6	81.7	34.1
	Medium	99.1	66.5	84.8	79.3	92.5	37.6
	High	99.8	88.3	93.8	97.7	98.6	49.1
	Wide	98.3	68.3	85.4	55.5	90.6	39.3
200	Low	98.4	70.1	89.5	79.2	90.1	42.5
	Medium	100.0	84.7	83.3	90.7	97.9	39.2
	High	100	96.1	99.5	100.0	99.5	54.9
	Wide	99.2	76.3	87.3	40.3	96.5	38.5
300	Low	99.2	74.9	87.0	81.1	95.8	44.4
	Medium	100.0	89.4	91.5	97.6	98.9	44.9
	High	100.0	97.7	100.0	100.0	99.8	57.9
	Wide	99.5	81.4	89.3	23.0	98.9	39.9
500	Low	99.3	80.9	87.8	88.9	99.0	60.4
	Medium	100.0	95.4	97.9	99.9	100.0	47.6
	High	100.0	99.0	100.0	100.0	100.0	65.4
	Wide	100.0	90.3	92.1	12.6	99.8	44.5

Note. N = sample size. h = communality.

to produce less change than decreasing from $\varepsilon = .01$ to .001 for small samples and the low communality condition and the mixed communality conditions.

Second, increasing communality tends to increase the similarity between global solutions at different levels of ε. For example, the percentages are 90.4%, 97.9%, and 99.5% between $\varepsilon = .02$ and .01 for Model M_4 at a sample size of 200. When there is much variation among manifest variable communalities, the percentages are close to those of the medium communality conditions with a few exceptions. The percentages of the wide conditions are much lower than others for Model M_{3b}. Third, increasing sample size tends to increase the similarity between global solutions at different levels of ε with the exceptions of the wide communality conditions of Model M_{3b}.

The percentages reported in Table 4 were obtained with a strict criterion: the congruence coefficients between all corresponding factors of two global solutions are larger than .98 in a simulated sample. We replicated the results with a less strict criterion: .92. It is regarded as reflecting a "good" correspondence between two factor loading matrices (MacCallum et al., 1999). The percentages become larger with this less strict criterion, but the general conclusions remain the same. The percentages corresponding to the criterion of .92 are included in the online appendices.

Comparing the global solution in simulated samples with the global solution in the population

We now compare the rotated factor loading matrix of the global solution in simulated samples with their respective population solutions to assess how population factors are recovered in samples. Note that these solutions are actually putative global solutions because we cannot rule out the possibility of another local solution with an even lower rotation criterion function. We focus on $\varepsilon = .01$ because it gives satisfactory results in a variety of conditions. We computed the congruence coefficients between columns of the sample factor loading matrix and the corresponding columns in the population factor loading matrix. A population factor is considered recovered if the congruence coefficient is larger than a certain value; a population model is considered recovered if the congruence coefficients of all factors are larger than the value.

Table 5 presents percentages of 1,000 simulated samples for which the population global solution is recovered in each condition for models M_{3a}, M_{3b}, and M_4. Two cutoff values of congruence coefficients are considered: .98 and .92. Three observations can be made. First, although model recovery improves as the cutoff value for congruence coefficients decreases, the general pattern regarding sample size and communalities appears to be similar for three cutoff values. Second, the percentages of model recovery increase as communality increases. For example, when the sample size is 60, the recovery rates for Model M_{3b} are 2.3%, 40.7%, and 83.3% at the low, medium, and high communality conditions at the cutoff value of .92. The corresponding recovery rate is 30.3% at the wide communality condition. Therefore, the recovery rate for the wide communality condition lies between those of the low communality condition and the medium communality condition. The influences of communality on recovery rates are similar for all four models, five

Table 5. Percentage of samples in which the population global solution is recovered by the sample global solution at $\varepsilon = .01$.

Conditions		Cutoff values					
		M_{3a}		M_{3b}		M_4	
N	*h*	.98	.92	.98	.92	.98	.92
60	Low	.0	.6	.0	2.3	.0	.0
	Medium	.4	24.5	.6	40.7	.0	8.6
	High	24.8	96.2	65.3	83.3	.1	83.3
	Wide	.0	9.6	.0	30.3	.0	5.8
100	Low	.0	3.8	.0	21.1	.0	.0
	Medium	1.6	63.5	23.8	64.4	.0	51.5
	High	65.2	99.5	96.5	96.9	12.2	98.5
	Wide	.3	22.8	1.0	55.9	.0	50.1
200	Low	.1	19.1	2.6	57.6	.0	6.9
	Medium	27.2	95.7	87.1	88.5	3.6	97.1
	High	98.6	100.0	100.0	100.0	81.4	99.7
	Wide	4.5	62.1	26.5	75.4	2.2	95.5
300	Low	1.3	39.6	28.9	73.1	.0	47.4
	Medium	59.6	99.7	97.5	97.5	35.0	99.5
	High	99.9	100.0	100.0	100.0	94.6	100.0
	Wide	17.0	82.3	59.1	81.7	27.5	98.7
500	Low	8.5	78.1	83.2	87.3	.2	92.3
	Medium	91.0	100.0	99.9	99.9	88.9	100.0
	High	100.0	100.0	100.0	100.0	99.9	100.0
	Wide	44.9	96.4	79.5	89.9	81.7	100.0

Note. N = sample size. h = communality. Two cutoff values (.98 and .92) for congruence coefficients are considered for assessing factor recovery.

levels of sample sizes, and three cutoff values with a few exceptions. Third, the recovery rate increases as sample size increases. For example, the recovery rates for Model M_{3b} increase to 21.2%, 64.4%, 96.6%, and 55.9% at the low, medium, high, and wide communality conditions at the cutoff value of .92 when the sample size increases to 100. Increasing sample size to 200, 300, and 500 produces even higher recovery rates.

The beneficial effects of larger sample sizes and higher communalities on factor recovery rates are expected. Larger sample sizes and higher communalities lead to smaller standard errors and more accurate parameter estimates, which increases the likelihood that the common factors are recovered in simulated samples. MacCallum, Widaman, Preacher, and Hong (2001) reported similar results with direct quartimin and target rotations. We focus on such effects for geomin rotation in this article. Local solutions are rare in direct quartimin and target rotation, but they frequently occur in geomin rotation. Comparing the population global solution with only sample global solutions may be insufficient, because a sample local solution may correspond to the population global solution.

Comparing local solutions in simulated samples with the global solution in the population

To further investigate the recovery performance of geomin rotation, we compare all solutions in a simulated sample with the global solution in the population. Figure 2 displays percentages of simulated samples in which population factors of Model M_{3b} are recovered by both the global solution and local solutions. Geomin rotation was conducted with $\varepsilon = .01$ and the cutoff value for congruence coefficients is .98.

Three observations can be made from Figure 2. First, higher communalities lead to higher recovery rates. Second, larger samples lead to higher recovery rates. The influence of the sample size on the recovery rates is more substantial for the low and wide communality conditions because of the ceiling effect in the high communality and medium communality conditions. The recovery rates are 100% in the high communality condition for samples larger than 100; the recovery rates are almost 100% in the medium communality conditions for samples larger than 200.

The last observation is the most important one. The gray bars indicate that the percentages of simulated samples in which the population global solutions are recovered by a sample local solution instead of the sample global solution. Such percentages are substantial in several conditions: they are 11.6% at a sample size 500 and the low communality condition, 9.3% at a sample size 100 and the medium communality condition, and 11% at a sample size of 60 and the high communality condition.

We also examined the percentages of the recovery rates of the population global solution by the sample global solutions and sample local solutions for the models M_{3a} and M_4. Furthermore, we examined the recovery performance with the cutoff value of .92 for the congruence coefficients. The percentages of recovering the global solution by a sample local solution in models M_{3a} and M_4 are less substantial than those for Model M_{3b}. A possible reason is that local solutions are less frequent

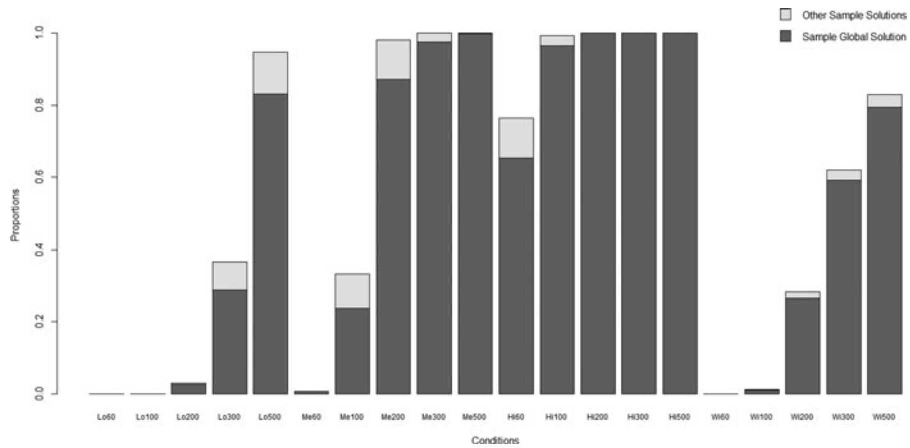

Figure 2. Percentage of replication samples whose sample solutions correspond to the population global solutions at $\varepsilon = .01$ for Model M_{3b}.

in the other three models than in model M_{3b} at $\varepsilon = .01$. Figures reporting such percentages for models M_{3a} and M_4 and both cutoff values for congruence coefficients are included in the online appendices.

The finding that the population global solution is recovered by a sample local solution instead of the sample global solution is important. If the factor analyst focuses on the sample global solution, one could completely miss the population global solution. We recommend examining all local solutions in addition to the global solution when interpreting rotated factor loading matrices.

Concluding comments

We investigated the occurrence, frequency, and nature of multiple local solutions in geomin rotation with empirical data and simulated data. We explored the influence of the geomin rotation parameter ε on local solutions. There are four major findings. First, the number of local solutions increases as the geomin rotation parameter ε decreases. Second, 100 random starts seem sufficient to obtain multiple solutions for $\varepsilon = .01$. Third, the global solution in the population may be recovered by a sample local solution instead of the sample global solution. Fourth, the population global solution is more likely to be recovered in a sample if communalities are high and sample size is large.

Different factor rotation methods are available to aid in the interpretation of EFA results. It is unrealistic to expect that a single rotation method will provide satisfactory results in all situations. Because EFA is a data-driven statistical procedure, factor analysts should consider and compare different rotation methods. In particular, geomin rotation should be included in such comparisons. Geomin rotation has two advantages. First, it can give satisfactory solutions when other rotation criteria fail to do so when the factor loading structure is complex, for example, Thurstone's 26-variable box data (Browne, 2001, Table 4). Second, geomin rotated solutions were shown to be comparable to target rotated solutions and CFA solutions for various models in simulation studies (Asparouhov & Muthén, 2009, Table 2). Both CFA and target rotation require factor analysts to provide additional information about the expected factor loading matrix, but geomin rotation does not require such information.

The use of geomin rotation requires the analyst to select a value for ε. We considered four values for ε: .02, .01, .001, and .0001. Our models included a two-factor model, two three-factor models, and a four-factor model. The number of local solution increases as ε decreases. In particular, when $\varepsilon = .0001$ and the EFA model included more than two factors, geomin rotation produced a different local solution from each random start. We recommend $\varepsilon = .01$ because (1) it effectively deals with the indeterminacy caused by zero loadings and (2) it does not change zero loadings too substantially. Although a smaller value like $\varepsilon = .001$ can also deal with the indeterminacy caused by zero loadings in a two-factor model, its advantage over $\varepsilon = .01$ is minimal. Browne et al. (2010) suggest that the default value for ε be .01 and a larger value be considered if the number of factors is four or greater. Asparouhov and Muthén (2009, p. 409) suggested that the default value for ε be .0001 for two factors, .001 for three factors, and .01 for four or more factors. Our recommendations for ε are similar to those made in Browne et al. (2010), but are larger than those made in Asparouhov and Muthén (2009) for two or three factors. Note that the global solutions obtained under $\varepsilon = .01$ were very similar to those obtained under smaller values of ε in our empirical data and simulated data.

It is essential to consider multiple random starts when conducting factor rotation using geomin rotation. The factor analyst could miss the global solution without repeating geomin rotation from multiple random starting values. In our experience, often, the global solution occurs less frequently than another local solution. Although we cannot guarantee that we find the global solution and all local solutions at any number of random starts, 100 random starts seem sufficient at $\varepsilon = .01$. For all four models we considered in the article, increasing the number of random starts to 1,000 does not return more local solutions. Smaller values of ε will require a larger number of random starts, however. When $\varepsilon = .001$ is used for the four factor model M_4, 30 random starts returned 6 solutions, 100 random starts returned 9 solutions, and 1,000 random starts returned 31 solutions.

When at least two solutions are produced from multiple random starts, we recommend that factor analysts examine all local solutions in addition to the global solution. As indicated by our simulated data, the population global solution is recovered by a sample local solution instead of the sample global solution. Ignoring local solutions would completely miss the population global solution. Rozeboom (1992) also argued for examining local solutions in factor rotation, because these local solutions offer opportunities of "catching interpretively provocative rotations of the input factors that might otherwise elude discovery" (p. 585). When the number of local solutions is large (e.g., 12), we need to make sure that all these solutions are "real" solutions. Because minimizing the geomin function can be difficult for some problems, the minimization algorithm can stop prematurely. Some of these "local" solutions may be due to the lack of convergence of the minimization algorithm. In addition, the occurrence rates of local solutions should be reasonably high. Thus, we recommend that factor analysts examine (1) the putative global solution, (2) several solutions whose criterion function values are close to that of the putative global solution, and (3) several solutions that occur most frequently. In addition, factor analysts can compute the congruence coefficients of all local solutions to seek for potential patterns. Results from other rotation methods should also be included in the comparison. We include an R function in the appendices to facilitate such comparisons.

The results that the population global solutions are more likely to be recovered in larger samples and with higher communalities agree with the results reported by MacCallum et al. (2001). Larger samples and higher communalities will produce smaller standard errors and more accurate estimates for rotated factor loadings. Thus, larger samples and higher communalities will benefit any factor rotation method. For geomin rotation, larger samples and higher communalities make it more likely that the population global solution is recovered by the sample global solution. As shown in Figure 2, increasing the sample size or the communalities makes it more likely that the sample global solution corresponds to the population global solution.

Because geomin rotation tends to have multiple local solutions, the bootstrap method becomes more difficult. For example, the goal may be to compute standard errors for the global solution in the original sample, but a local solution rather than the global solution in a bootstrap sample may correspond to a global solution in the original sample. One possible solution is to select a solution in a bootstrap sample that aligns best with the global solution in the original sample: the selected solution could be either a local solution or a global solution in the bootstrap sample.

The phenomenon of local solutions is not unique to geomin rotation. Infomax rotation and minimum entropy rotation also tend to produce local solutions (Browne, 2001, Table 4). Gebhardt (1968) even described multiple local solutions for rotation criteria like varimax or quartimax, but occurrences of local solutions with varimax and quartimax rotation in empirical studies have not been reported. Note that the functional forms of geomin, infomax, and minimum entropy are more complex than those of varimax and quartimax. The geomin criterion involves taking the m-root of the product of m terms; infomax and minimum entropy involve taking log functions. In contrast, varimax and quartimax (direct oblimin) involve only fourth-order polynomials. The more complex functional forms of geomin, infomax, and minimum entropy are likely to contribute to the local solution phenomenon of these rotation methods. In addition, the current study was conducted under conditions similar to conventional applications of factor analysis in the social and behavioral sciences. Future research efforts need to focus on factor analysis with larger problems (possibly thousands of variables and hundreds of factors) in this new era of big data. Local solutions are very likely to occur in these situations.

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