From Modeling Long-Term Growth to Short-Term Fluctuations: Differential Equation Modeling Is the Language of Change

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1 Introduction

Language is an integral part of expressing ideas, so much so that the statistical language(s) we understand may affect our ability to formulate ideas. One's initial statistics class consists of exposure to a new language; familiar words take on new meaning, mathematical symbols are used to abbreviate entire paragraphs, and you even learn a little Greek. Learning other dialects, for example structural equation modeling (SEM) diagrams, expands one's ability to posit and understand statistical models. The numerous simultaneous regressions occurring in many SEM diagrams would be difficult to understand as a list of equations, but these equations become readily accessible when expressed in the language of SEM diagrams. The representation of regression in diagram form allows for the formulation and understanding of new ideas. Differential equations, and their component derivatives,

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constitute a language that is less often used in the social sciences; the few clear exceptions like the Hessian matrix and calculation of the minima or maxima of functions are in the vernacular of relatively few social scientists. Like learning the language of SEM, learning the language of derivatives has the potential to change the way we understand models with which we are familiar, and opens us to new ways of formulating ideas.

In this chapter we present the idea that differential equation modeling is the language of change. The meaning in this statement is twofold. In the literal sense derivatives express the change in variables with respect to each other; differential equation models—models that include derivatives—are models that express the relations between the states of variables and how variables are changing. Derivatives and differential equations provide a language that gives a framework for precisely describing change. But this approach also provides a different way of understanding many of the models of change that are currently being used in research; by providing a unifying framework, differential equations have the potential to help in identifying models that have been overlooked and therefore can identify unexplored questions. By providing a means to alter how questions about change are being asked, differential equation models constitute a language that could lead to changes in the kinds of research being done.

This chapter begins by considering differential equations and derivatives in the context of something familiar—ordinary linear regression. As the new language is introduced, the chapter expands into other familiar models including hierarchical linear models (HLMs) and latent growth curve models (LGCMs). These sections introduce the derivative language framework as literally being a language for describing change. We then consider the application of derivatives to the modeling of intraindividual observations. The language framework is used to extend the idea of differential equation modeling as the language of change so as to introduce methods and models that are likely to be unfamiliar to many readers. Three differential equation models will be presented, each of which provides cutting edge questions that can be addressed using social science data.

2 Regression

Whether made explicit or not, early in statistics classes students are introduced to the idea that mathematics can be used to address whether one variable is related to another, and more specifically that the change in one variable can be related to changes in another variable. This idea often begins with the case of relating central tendency to group membership (i.e., t-tests), but becomes more general with the introduction of ordinary linear regression. This idea gets a formal mathematical representation,

$$y = \beta_0 + \beta_1 x + \epsilon, \tag{1}$$

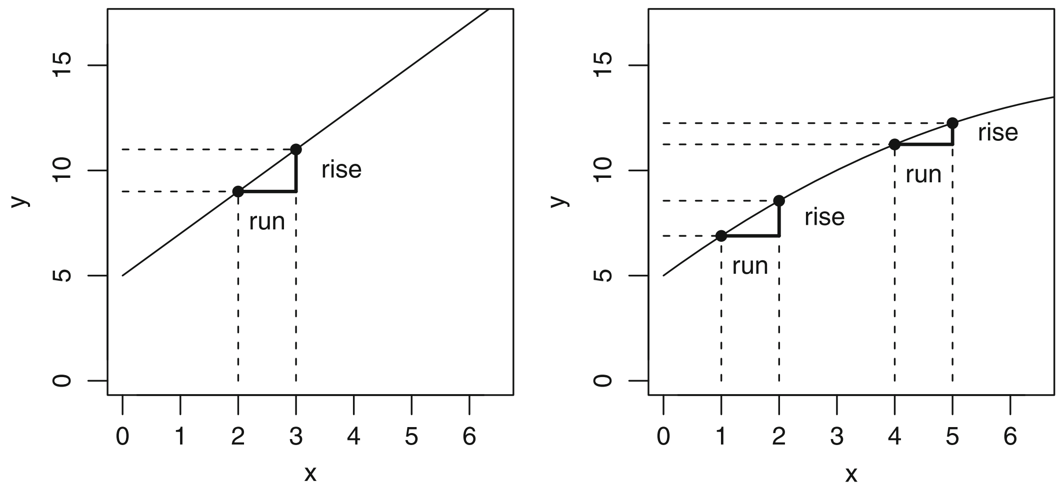

Fig. 1 The figure (*left*) depicts a trajectory with a constant first derivative, and consequently a second derivative equal to zero. The figure (*right*) depicts a trajectory where there is a change in the first derivative with respect to time; that is, the second derivative is nonzero

where β_0 represents the value of a variable y when $x = 0$, β_1 expresses how changes in a variable x are related to changes in y, and ϵ represents error. This seemingly simple equation allows for significant amounts of statistical language to be taught, including keywords like *intercept* and *slope*, *statistic* and *parameter*. Interpretation of β_0 and β_1 also becomes an important exercise, at which point figures such as Fig. 1 (left) may be used. In Fig. 1 it has been assumed that we are working with the equation $y = 5 + 2x$, and therefore $\beta_0 = 5$ is the value of y when $x = 0$. A series of points (joined with a line) can then be drawn, substituting values $x = 1, 2, 3\ldots$ and solving for y. Earlier in your education, you may have been given a line and asked: "what is the rise over the run?" Said another way, this question asks how much of a change in y coincides with a specific amount of change in x (one unit); that is, what is β_1?

"Rise over run" can be equivalently expressed as "the change in y with respect to the change in x." In mathematics this is frequently expressed as $\frac{dy}{dx}$; this is the *first derivative* of y with respect to x. Instead of writing β_1 it would be equally appropriate to write

$$y = y_0 + \left(\frac{dy}{dx}\right)x + \epsilon, \tag{2}$$

where y_0 is the *zeroth derivative* which is the value of y at $x = 0$. This form of the regression equation, in the authors' experience, seems to appear rarely in introductory statistics texts. One likely reason is that Eq. (2) may be perceived as more complex than Eq. (1), even though these equations are equivalent. Another reason may be that the equivalence of β_1 and $\frac{dy}{dx}$ is thought to be commonly understood, so stating this explicitly in equations is considered unnecessary.

Whatever the reason, by not being explicit that β_1 is a derivative, useful language has been set aside. Consider the quadratic model,

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \epsilon. \tag{3}$$

In presentations of this model, it is not unusual to hear that β_2 is difficult to directly interpret; so much so that efforts have been undertaken to reparameterize β_0, β_1, and β_2 so as to make the parameters more readily interpretable (Cudeck and du Toit 2002). Making the derivatives explicit, Eq. (3) is equivalent to

$$y = y_0 + \left(\frac{dy}{dx}\right) x + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right) x^2 + \epsilon. \tag{4}$$

β_2 in itself is difficult to understand, but twice this quantity is equal to the *second derivative*, $\frac{d^2y}{dx^2}$. The second derivative expresses how the first derivative $\frac{dy}{dx}$ is changing with respect to changes in x. That is, twice the quadratic parameter β_2 conveys precisely how quickly the slope (first derivative) is changing for every unit change in x.

A person's score y based on Eq. (4) depends on three things: (a) the score at $x = 0$, that is, the zeroth derivative, (b) the rate at which scores change with respect to x (slope or first derivative) at $x = 0$, and (c) how the slope changes with respect to x (second derivative). If x represents time and y position, derivatives can be discussed drawing on the common experience of traveling in a vehicle. The zeroth derivative is the position, or *level* in the case of a construct, at some point in time. The first derivative, or change in position with respect to time, represents *velocity* (speed in a particular direction). The second derivative, or change in velocity with respect to time, corresponds to *acceleration* (positive or negative).

Early introduction of derivative language has the potential to provide a unifying framework for understanding many models of change. Extending Eq. (4) to include predictors of the estimated derivatives (parameters), as is often done in Hierarchical Linear Modeling (HLM) or Multilevel Modeling (MLM), the language of derivatives gives another way to understand the hypotheses being tested. Consider the equations

$$y_{ti} = \beta_{0i} + \beta_{1i} T_{ti} + \beta_{2i} T_{ti}^2 + \epsilon_{ti} \tag{5}$$

$$\begin{aligned} \beta_{0i} &= \gamma_{00} + \gamma_{01} Z_i + u_{0i} \\ \beta_{1i} &= \gamma_{10} + \gamma_{11} Z_i + u_{1i} \\ \beta_{2i} &= \gamma_{20} + \gamma_{21} Z_i + u_{2i}, \end{aligned}$$

where the dependent variable y_{ti} is measured at multiple times t for each individual i. The effect of the predictor *time* (T) allows for a linear relationship (β_{1i}) with time and the possibility that the slope (β_{1i} at $T = 0$) may change with respect to time

(β_{2i}). Furthermore, each individual may have a different β_0, β_1, and/or β_2; it is hypothesized that individual differences in these parameters are related to a person's trait Z_i.

Rewriting the equations expressed in Eq. (5) using derivative notation

$$y_{0i} = \gamma_{00} + \gamma_{01} Z_i + u_{0i} \tag{6}$$

$$\left(\frac{dy}{dT}\right)_i = \gamma_{10} + \gamma_{11} Z_i + u_{1i} \tag{7}$$

$$\frac{1}{2}\left(\frac{d^2y}{dT^2}\right)_i = \gamma_{20} + \gamma_{21} Z_i + u_{2i}. \tag{8}$$

These equations express that in HLM/MLM one is examining the relations between the level (zeroth derivative) of a trait Z and derivatives expressing different aspects of how the dependent variable y is changing with respect to time T. In Eq. (6), γ_{01} posits a relation[1] between the zeroth derivative of the dependent variable y and the trait Z. In Eq. (7), γ_{11} relates the first derivative of the dependent variable to the trait. Finally, in Eq. (8), γ_{21} relates the second derivative of y to the trait. These three equations ask qualitatively different questions. The first asks whether Z is related to the level of y at $T = 0$; that is, a *level–level* relation. The second asks whether Z is related to the velocity of y at $T = 0$; that is, a *level–velocity* relation. The third asks whether Z is related to changes in velocity; that is, a *level–acceleration* relation.

Equations (5) and (6) through (8) show HLM/MLM as a series of differential equations. Looking at these equations, and considering the relations between level, velocity, and acceleration gives another way to understand this model in terms of level–level questions (Eq. (6)), level–velocity questions (Eq. (7)), and level–acceleration questions (Eq. (8)). Examining other models, and the relationships between derivatives that are being modeled, provides a way to organize the similarities and differences across a wide range of models of change. In the next section we examine the LGCM, which can relate both similar and different pairs of derivatives relative to HLM and therefore ask both similar and different questions about change. The decision to use one over the other should be driven by constraints such as the structure of the data collected, for example HLMs/MLMs can handle individuals with variations in sampling interval more readily than LGCMs; conversely, LGCMs can handle multiple dependent variables simultaneously. The decision to use one over the other should not be driven by the perception that these models are necessarily addressing different questions, as in some cases the questions being asked are very similar.

[1]This relation could be expanded to indicate that parameters such as γ_{11} express the change in an individual's first derivative $d\left(\frac{dy}{dT}\right)_i$ (numerator) divided by the change in the trait dZ_i.

3 Latent Growth Curve Model

We posit an example where we consider the effect of Stress on Negative Affect measured across the last 4 weeks of a semester in a hypothetical sample of undergraduate students. A LGCM is posited, as in Fig. 2, such that changes in stress result in changes to negative affect; a unidirectional relationship from stress to negative affect has been posited only to simplify discussion and is not based on theory. Typically the latent variables would be labeled "Intercept," "Slope," and "Quadratic," and the paths to the observed variables would all be fixed such that the latent variables would correspond to the names they were given. It may not be clear, however, what a quadratic–quadratic relationship implies. In Fig. 2, therefore, the labels have been changed to "Level," "Velocity," and "Acceleration" to reflect that the zeroth, first, and second derivatives, with respect to time, are being estimated; to accomplish this, only the loadings of the quadratic factor are changed, and these are merely multiplied by 1/2 as in Eq. (4).

There are many possible paths that could be drawn from stress to negative affect (paths A through I). Some of these paths are familiar, such as paths A, B, and C which ask level–level, level–velocity, and level–acceleration questions as in HLM.

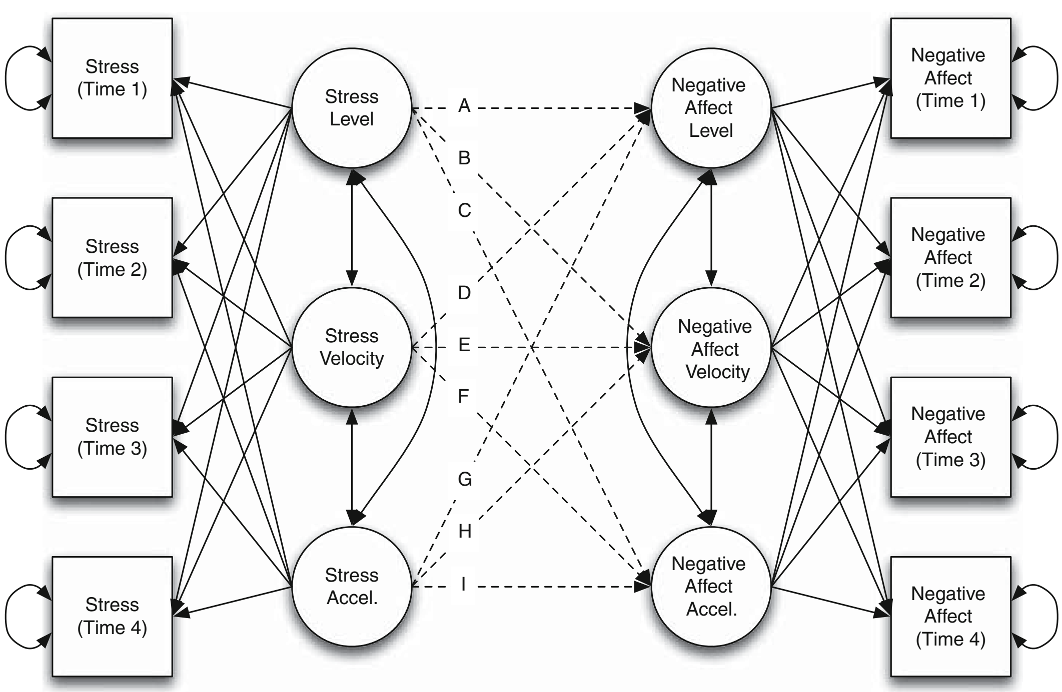

Fig. 2 A latent growth curve model which, with slightly modified fixed paths, expresses the level, velocity, and acceleration in stress and negative affect as latent variables. Many possible relationships between the stress and negative affect derivatives could be considered (paths A through I). It should be noted that causal interpretations of this model may not make sense, depending on how time has been coded. When Time = 1 is used as the initial time, paths D and G suggest that later changes over the four observations could alter one's initial level of Negative Affect (i.e., time travel)

LGCM allows for questions similar to HLM to be addressed, but also a multitude of additional questions (paths D through I). What then is meant when one states that stress and negative affect are related? Should we posit all paths?

When changed to level, velocity, and acceleration, it may be possible to argue that there are only a few paths that are of theoretical interest. First one must consider the dependent variable (negative affect): Do we wish to predict a person's level of negative affect at $T = 0$? Do we wish to predict a person's velocity at $T = 0$? or Do we wish to predict whether a person's trajectory of negative affect is changing—what traits are related to a person departing from their initial trajectory? The third question is one about the acceleration in negative affect (second derivative; paths C, F, and I). Turning to the predictor, what is it about stress that is related to changes in the velocity of negative affect (i.e., acceleration)? Is one's level of stress related to changes in the velocity of negative affect? Or, is it that increases in stress are related to changes in the velocity of negative affect? Or, is it the fact that one's stress is not just increasing, but increasing at a faster rate, that is related to changes in the velocity of negative affect? These three questions are qualitatively different, and the presence of any one relation does not necessarily imply anything about the presence or lack of the other two relations.

The question "Is stress related to negative affect?" is too simple, as even limiting ourselves to a unidirectional case implies nine possible relations as in Fig. 2 (paths A through I), or even more if one considers higher order derivatives. Even narrowing our interest to what is related to a change in the velocity of negative affect, there are still multiple possibilities to consider (paths C, F, and I), each of which addresses a qualitatively different question. What is it that most directly causes one's negative affect trajectory to curve (accelerate) in a negative direction—is it the specific level of one's stress, the fact that one's stress has been increasing for several weeks, or that one's stress level is increasing at a faster and faster rate? Whatever one's response, the language of derivatives allows us to more clearly highlight the questions addressed by the LGCM (as opposed to considering quadratic–quadratic relations). The common language between this section and the previous section also highlights that LGCM has the potential to address many of the questions addressed by HLM (see Bauer 2003; MacCallum et al. 1997).

4 Derivative Language Framework

In this chapter we have introduced the language of derivatives in a manner intended for a broad audience, and without requiring an introduction to calculus. Just the realization that many common models contain parameters that express the change in one variable with respect to another (derivatives), and labeling them as level, velocity, and acceleration, has the potential to provide researchers a novel language framework for understanding a variety of models. There are at least four consequences of adopting the language of differential equations and derivatives.

Table 1 Summary of derivatives related in several common methods for the analysis of change

		Construct 2		
		y	dy/dt	d^2y/dt^2
	x	Correlation[a]		
		Ordinary Regression[a]		
		SEM[a]		
		HLM/MLM		
		LGCM		
		GLLA/GOLD/LDE		
Construct 1	dx/dt	HLM/MLM		
		LGCM	LGCM/PPM	
		LCS	LCS	
		LPM/CLPM		
		GLLA/GOLD/LDE	GLLA/GOLD/LDE	
	d^2x/dt^2	HLM/MLM[b]		
		LGCM[b]	LGCM[b]	LGCM[b]
		LCS[b]	LCS[b]	LCS[b]
		GLLA/GOLD/LDE	GLLA/GOLD/LDE	GLLA/GOLD/LDE

SEM structural equation modeling, *HLM/MLM* Hierarchical Linear Modeling/Multilevel Modeling, *PPM* Parallel Process Modeling, *LGCM* latent growth curve modeling, *LCS* Latent Change Scores, *LPM/CLPM* lagged panel modeling/cross-lagged panel modeling, *GLLA/GOLD/LDE* Generalized Local Linear Approximation, Generalized Orthogonal Local Derivatives, Latent Differential Equations

[a]Many, but not all applications

[b]Applications corresponding to this relationship are unusual

First, by thinking about the possible ways derivatives can be related—level–acceleration relations, velocity–velocity relations, level–level relations, etc.—there is a relatively limited number of combinations that are possible (nine, unless higher order derivatives are considered). Rather than continue to present students an ever-increasing number of models and methods for describing change, a matrix of derivative relations could be introduced (e.g., Table 1). Each method/model would fall into one or more of the finite number of combinations. The differences between all methods/models that allow for level–acceleration questions to be addressed could then be compared and contrasted. From the authors' perspective, some of the key differences are the kind of data to which a particular method/model is typically applied, and the time scale over which derivatives are being estimated (Deboeck et al. submitted).

Second, using this language framework allows for the presentation of a theory–method Rosetta stone as in Table 2. Using level, velocity, and acceleration would allow researchers to be much more specific with regard to theories of change. But as these words are directly related to the zeroth, first, and second derivatives, the mathematical interpretation of these words is very precise. The challenging endeavor of translating theory into mathematics can then be made much more precise. Moreover, in areas where theory is rich, use of this language may drive the development of new, more appropriate models.

Table 2 Summary of several equivalent ways to express the zeroth through second derivatives

Characteristic of scores	Derivative	Name	Graphical depiction	Notation
Score at some time	0th	Level	Single point	y
Rate at which level is changing	1st	Velocity	Straight line	dy/dt
Rate at which velocity is changing	2nd	Acceleration	Curved line	d^2y/dt^2

Third, this framework would allow for more detailed and accurate interpretation of results. Putting into words the differences between Eqs. (6) through (8), or paths A through I in Fig. 2, may be challenging. By highlighting that the parameters and latent variables can take on names associated with change—level, velocity, acceleration—may allow the hypotheses being tested to be more readily put into words.

Fourth, this new framework provides a structure that allows for the understanding of new methodology relative to well-known models. The following section introduces three differential equations that can be used to model the complex, nonlinear changes in studies of intraindividual variability. These models will be introduced relative to the more familiar LGCM. In introducing these newer methods, we highlight some new questions that become accessible using the derivative language framework.

5 Modeling Intraindividual Observations

The collection of repeated, intraindividual measurements on psychological and behavioral variables presents a new challenge for modeling. To provide an example of these challenges, we take as a motivating example daily measurements of positive and negative affect from the Notre Dame Study of Health & Well-being. Figure 3 shows estimates of positive affect measured over time, representing a sample of everyday positive affect (i.e., not following any particular stressor). One way to model these data would be to consider an HLM or LGCM, which would give some impression of the overall trajectory. This trajectory might be related to changes in season, or other macrotemporal changes occurring in the participants' lives but not directly related to the daily regulation of emotions. Moreover, HLM, LGCM, and many other models designate the variation around the overall trajectory as error, when in fact it might be the case that characteristics of this variability are related to important differences between individuals, such as resiliency.

As introduced in the previous section, and depicted in Fig. 4a, the latent variables of an LGCM can be used to estimate the *level* at $T = 0$, the *velocity* at $T = 0$, and the *acceleration* across a series of observations. This model is closely related to Latent Differential Equation Modeling (LDE; Boker et al. 2004), a method for modeling the rich, complex nonlinearity of intraindividual variability. Despite the differences the names may convey, these methods have many similarities: both can estimate the same derivatives and allow for the same relations between derivatives

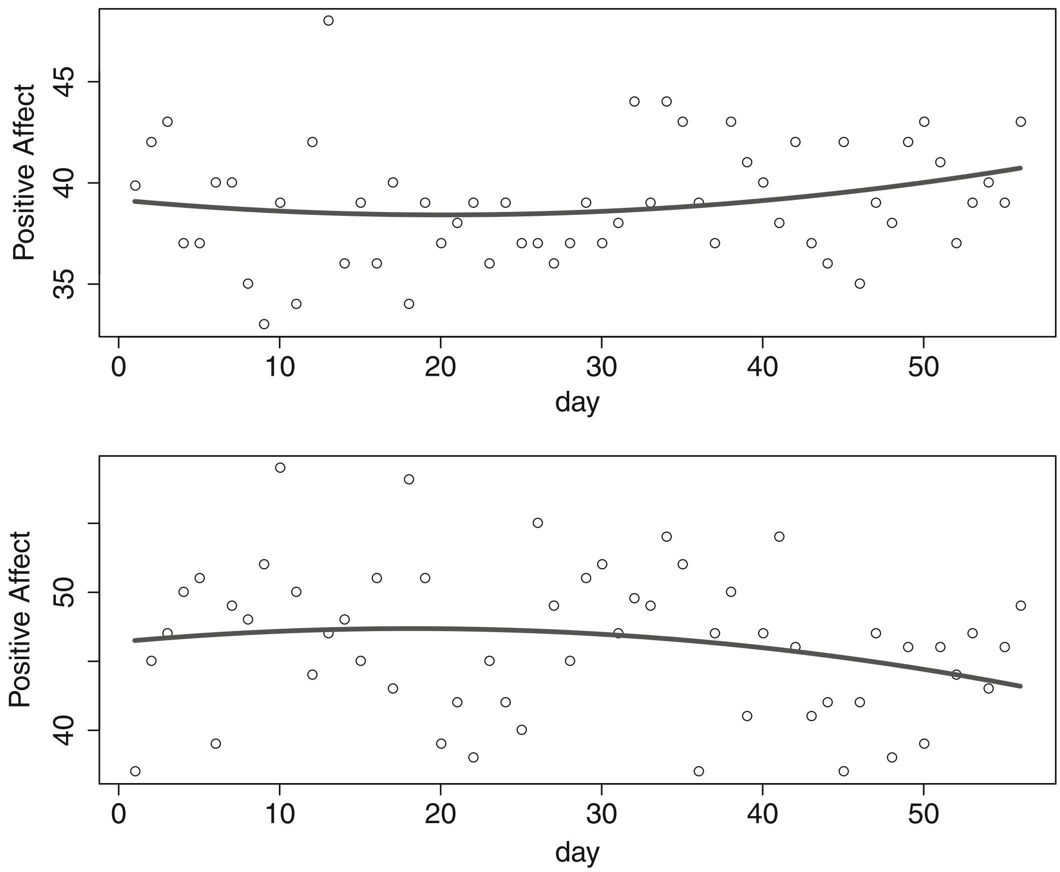

Fig. 3 Plots of positive affect measured over the course of 56 days. The plots contain the data from two different older adults. The *gray lines* are based on the estimated values of individual-specific quadratic regression models. Positive affect was measured using the PANAS (Watson et al. 1988) administered to older adults in paper and pencil daily diary self-report

to be examined (e.g., level–acceleration relations). The key difference between the two methods is the time scale over which they are applied; rather than estimate derivatives over the entire period of observations as in Fig. 4a, LDE estimates derivatives over the course of just a few observations as in Fig. 4b. The model in Fig. 4b may appear an impossible model to fit, but this is not the case once the data are reorganized into what is called an *embedded matrix*. In a manner akin to the depiction in Fig. 4c, one can rearrange data such that each row of a matrix consists of a subset of a longer time series. For example, given a time series $y = y_1, y_2, y_3, \ldots, y_t$ one can create an embedded matrix

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & y_3 & y_4 & y_5 \\ y_3 & y_4 & y_5 & y_6 \\ \vdots & \vdots & \vdots & \vdots \\ y_{t-3} & y_{t-2} & y_{t-1} & y_t \end{bmatrix} \tag{9}$$

a

b

c

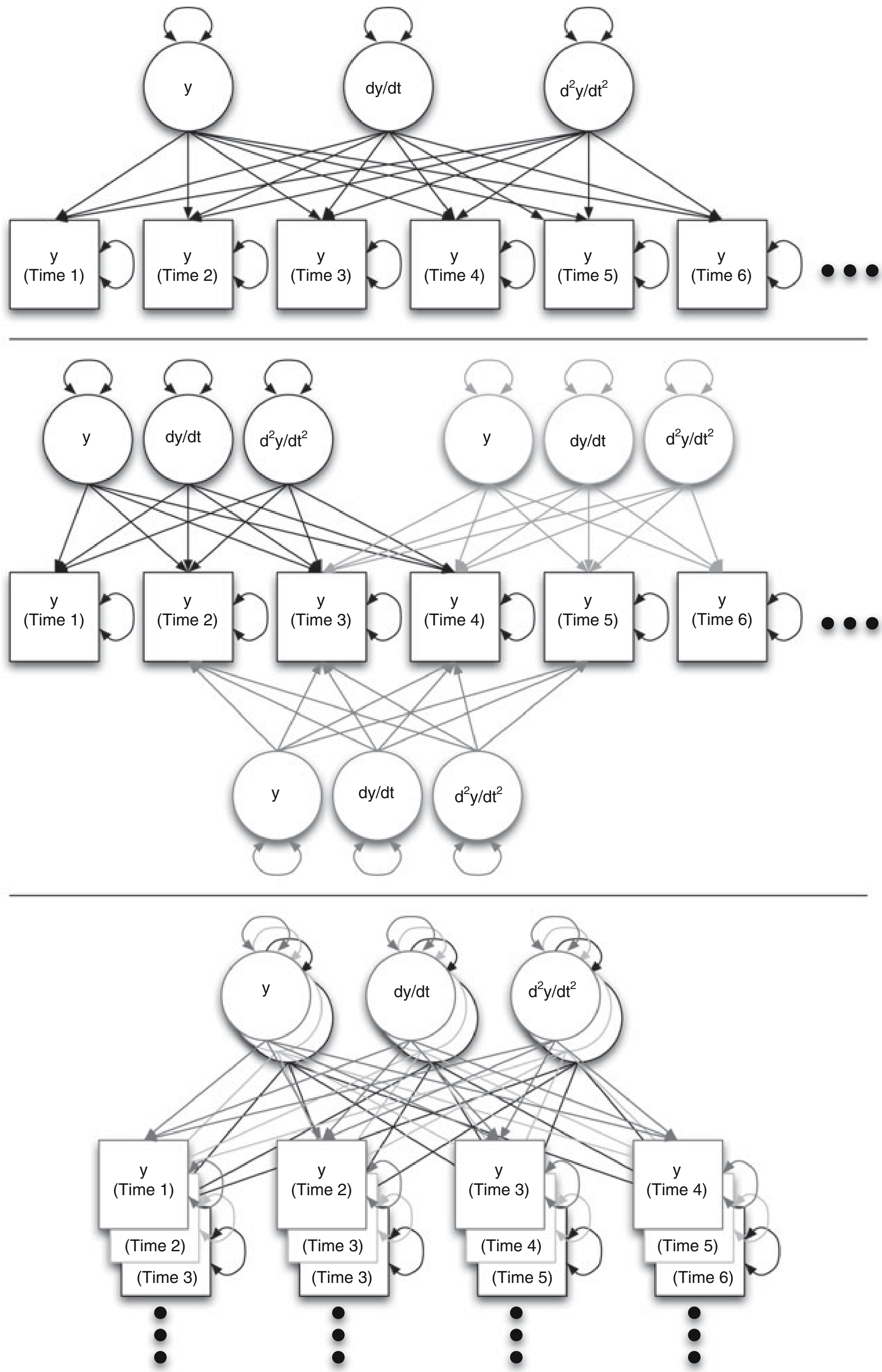

Fig. 4 Three differing versions of a latent growth curve model. (**a**) The LGCM can be applied to entire time series, providing a single estimate of the derivatives. (**b**) Many small LGCMs can be applied to a time series to generate estimates of derivatives at many different times across the series. (**c**) A revisualization of model (**b**) where the small LGCMs have been stacked. This both aids in estimation and allows one to think about the creation of an embedded data matrix which involves arranging data much as the observed variables have been stacked in this figure

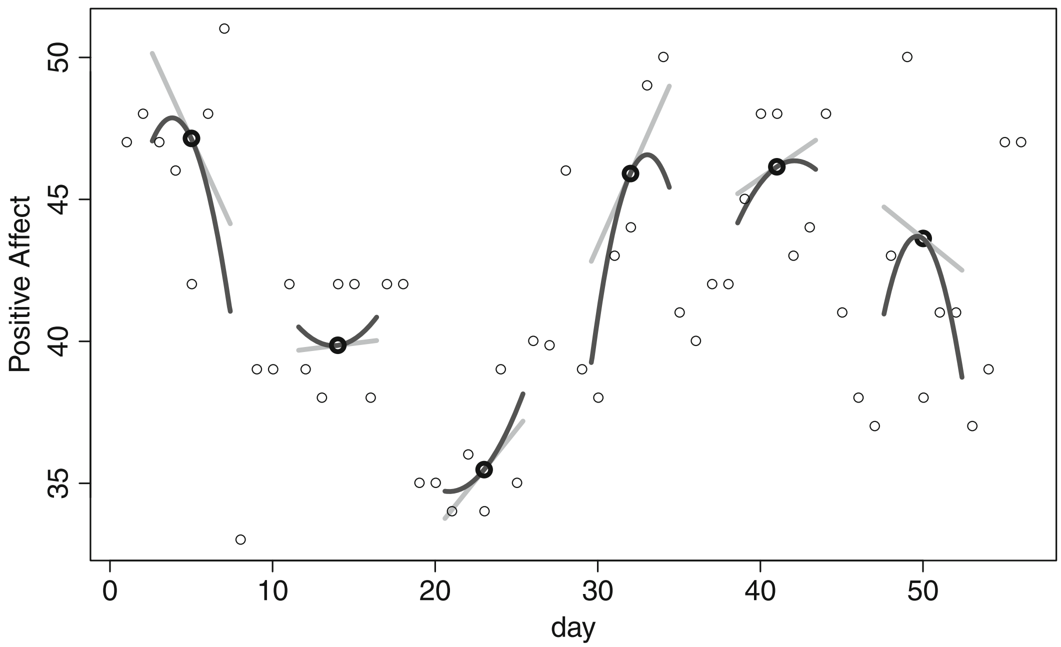

Fig. 5 By applying the latent growth curve model to short series of observations (*small circles*), the level (*bold*, *larger circles*), velocity (*straight*, *light gray lines*), and acceleration (*curved*, *dark gray lines*) can be estimated at many different times across a time series. The *gray acceleration lines* have been drawn such that they represent the slope that is expected at some time before/after the point at which the estimate was created; for example, for the estimate at day 23, the acceleration is such that a velocity of nearly zero would be estimated for day 20–21, and a relatively steep positive slope would be estimated for day 25–26. Confidence intervals exist for the derivative estimates, but have not been displayed to simplify the figure

where the first three rows of the matrix match the observed variable labels in Fig. 4c. Readers interested in more specifics about applying LDE are referred to Boker et al. (2004) and Deboeck (2011).

What does changing the time scale of derivative estimation buy us? In a LGCM we would have single estimates of level, velocity, and acceleration. In LDE we have estimates of level, velocity, and acceleration across an individual's time series, and therefore can observe how these values are changing, as in Fig. 5. In this figure, the observations (small circles) are used to estimate the level (bold circles), velocity (straight, light gray lines), and acceleration (curved, dark gray lines). The challenge lies in finding predictors of the derivatives across time. If the language of derivatives is applied to theoretical models, one could then translate theory into testable models to address research questions. Alternatively, data can be explored by examining predictors of different derivatives. As with the LGCM presented earlier, there are many possible derivative relations that could be considered (Fig. 2). Being precise as to how the level, velocity, and acceleration of variables affect each other over the span of a few days, however, is largely unexplored territory for many fields of study.

The following sections introduce three differential equation models. The models can be implemented in a variety of ways, including LDE (Boker et al. 2004),

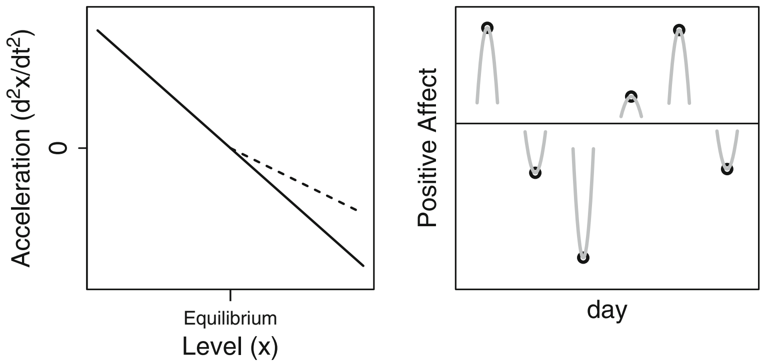

Fig. 6 A linear regression of level on acceleration (*solid line*, *left*) describes a model that implies that as one's level on a construct (*circles*, *right*) gets far from equilibrium (*horizontal line*, *right*) there is an acceleration (*gray curves*, *right*) in the direction of the equilibrium. As regressing level on acceleration is a linear relationship (*solid line*, *left*), one can borrow ideas from linear regression; for example, using piecewise regression one could allow the level–acceleration relation to be different when one is above equilibrium compared to when one is below equilibrium (*solid line* below equilibrium, *dashed line* above equilibrium)

Generalized Orthogonal Linear Derivative Estimates (Deboeck 2010), the Exact Discrete Model (Oud and Jansen 2000; Voelkle et al. 2012) and Generalized Local Linear Approximation (Boker et al. 2008; Boker and Nesselroade 2002). Through the three models that we present, we explore a few ideas of how relating derivatives may give some insight into certain processes. These models have not been applied in a wide range of contexts, so for many areas of the social sciences these are examples of how differential equations can provide a language to address new questions about change.

5.1 Model 1

The first model considers only a single variable—positive affect. As with the LGCM, we focus on the questions: What leads to changes in the trajectory of positive affect? and What is related to positive affect acceleration? But now these questions are being considered in the context of having made multiple derivative estimates over a time series as in Fig. 5. One way to model these data would be to posit additional variables, the derivatives of which might explain positive affect acceleration. Another option is to consider how the level, velocity, and acceleration estimates of positive affect might be related to each other.

For example, consider the linear relation (solid line) that has been drawn between the acceleration and level of positive affect in Fig. 6 (left). The interpretation of this relation is interesting, as when the level of positive affect is high there is negative acceleration; conversely, when positive affect is low, there is positive acceleration. This is depicted in another way in Fig. 6 (right). Such a relation would suggest that if one were near some average or typical level of positive affect (horizontal line) there might not be much change. But if one develops a high value of positive affect, negative acceleration is expected; the slope is changing so that the upward trajectory is not maintained, and eventually a negative trajectory will occur. The inverse is true for a low positive affect score.

This is one possible model of homeostasis or self-regulation. There is a typical state, or *equilibrium*, around which affect is expected to vary. Moreover, when affect is displaced far from equilibrium in either direction, there is a relation with an acceleration in the opposite direction, suggesting a change in slopes that would result in changes towards equilibrium. This model does not specify the mechanisms that lead to self-regulation, but it may be useful for characterizing how quickly individuals move towards and away from equilibrium; that is, do the gray acceleration curves in Fig. 6 (right) have a very steep or very shallow u-shape? The relationship in Fig. 6 can be written as the differential equation

$$\frac{d^2x}{dt^2} = \beta x + \epsilon, \tag{10}$$

which expresses that the second derivative (acceleration) is related to the zeroth derivative (level) times β plus error ϵ. The relationship β—which expresses how changes in the level of the self-regulating variable are related to changes in the acceleration of the same variable—is related to how quickly people return to, and move away from, their equilibrium state. For examples of papers implementing this model, see Bisconti et al. (2006), Boker and Laurenceau (2005), Montpetit et al. (2010), and Nicholson et al. (2011).

As the relationship in Fig. 6 and Eq. (10) consists of a linear regression, one can draw on familiarity with regression to inform how this model could be modified for different contexts. For example, the present model assumes the same relationship between acceleration and level regardless of whether one is above or below one's equilibrium. Perhaps it is expected that the level–acceleration relation above equilibrium is different than when it is below equilibrium; the rate at which one returns to equilibrium differs above and below one's equilibrium (see Fig. 6, left). This would correspond to a different slope above and below equilibrium, as depicted with the solid line below equilibrium and the dashed line above equilibrium. One could allow for the differing slopes using piecewise regression, using the equation

$$\frac{d^2x}{dt^2} = \beta_1 x + \beta_2 x_2 + \epsilon. \tag{11}$$

where x_2 is coded to be zero for negative values of x, and x_2 would be equal to x for positive values of x.

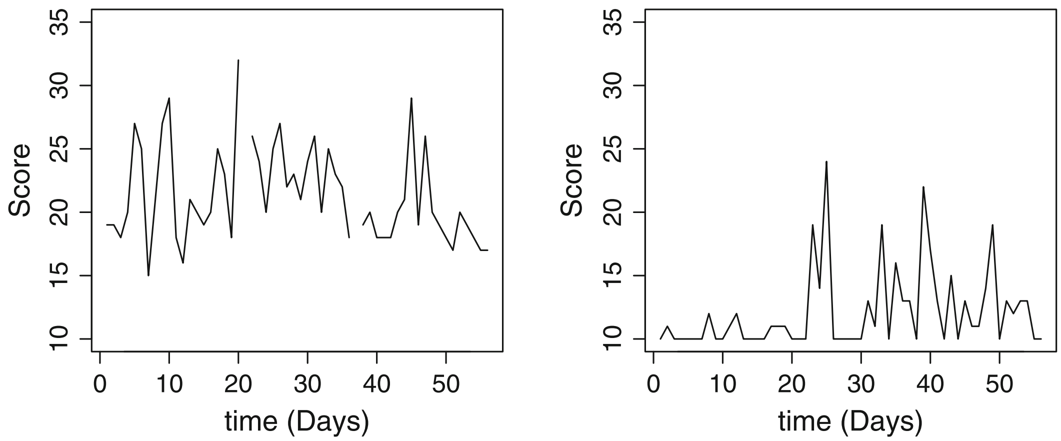

Fig. 7 Simulated plots of differing types of trajectories observed when measuring negative affect in a sample of older adults

5.2 Model 2

Models of self-regulation may be reasonable first approximations in many contexts; as in any domain, however, application of these models will require refinement. Figure 7 shows plots that are not atypical of what is observed when examining negative affect in older adults (Ram 2011). Some of the data patterns might be reasonably characterized using a model of self-regulation; there appears to be some equilibrium state around which an individual varies (Fig. 7, left). It is not unusual, however, to also observe patterns such as in Fig. 7 (right). In this figure, there appears to be a floor effect. Initially, one may expect this is due to a measurement problem, which could be solved by including items that would be more commonly endorsed. Attempts to take such a step, however, appear to mitigate but do not fully alleviate the presence of floor effects (Deboeck and Bergeman 2013). When examining negative affect, there appears to be a large proportion of individuals who do not follow a self-regulation-like model, but rather appear to register very low levels of negative affect that on occasion will increase in response to events.

One idea that has been proposed for modeling these data is the differential equation model

$$\frac{dx}{dt} = \beta x + \epsilon, \tag{12}$$

where the slope between days (first derivative) is related to the level of negative affect plus error (Deboeck and Bergeman 2013). Unlike other models, the errors in this model are assumed to consist of only positive values; for example, ϵ could follow a gamma distribution. If the errors are all positive, and the values of x are all positive, the value of β is required to be negative, otherwise scores would be required to monotonically increase for the duration of the study.

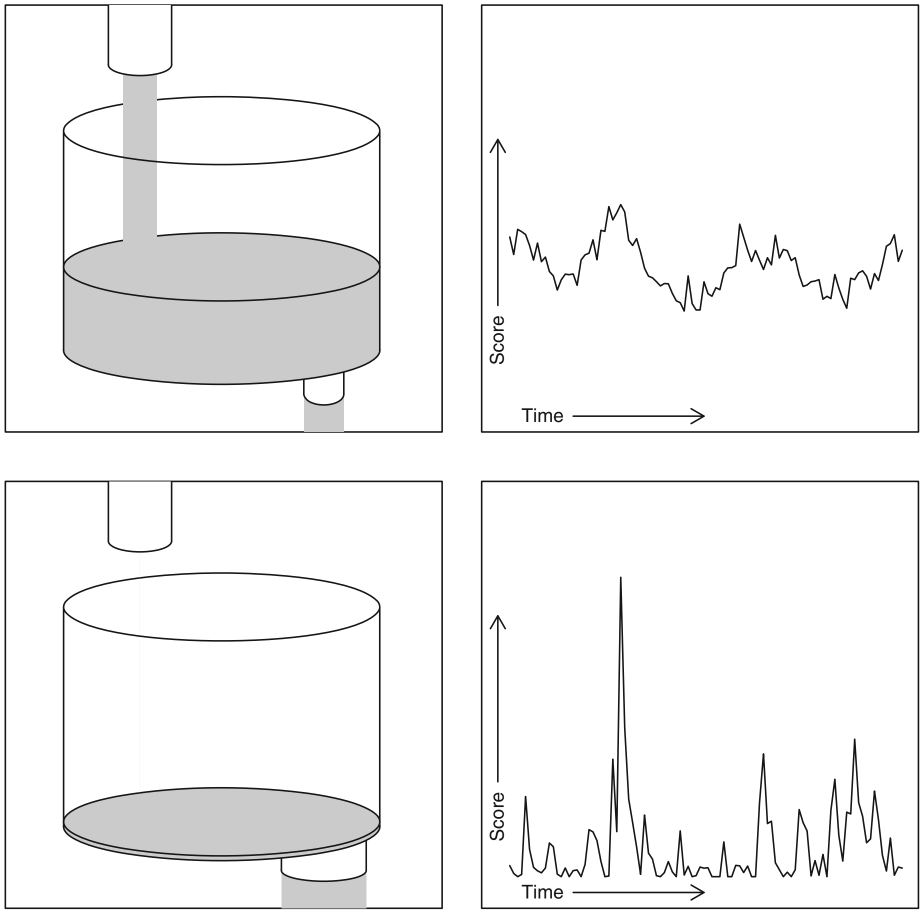

Fig. 8 Examples of the trajectories (*right column*) that occur from recording the height of the liquid in a simulation of two reservoirs (*left column*). In the reservoir in the *top row*, the rate of liquid (*gray*) inflow and outflow are approximately balanced; in this case, the reservoir always has liquid, although its level fluctuates around an equilibrium-like value. In the reservoir in the *bottom row*, the outflow is faster than the inflow; consequently, the trajectory often approaches the minimum value (empty reservoir) except when a large input event occurs

A metaphor for the behavior of this model is that of a reservoir, as in Fig. 8 (left column). The negative affect reported on any given day corresponds to the height (level) of the liquid in this reservoir. Above the reservoir is a pipe that adds liquid to the reservoir, increasing the height of the liquid; the added liquid corresponds to any events perceived as increasing one's negative affect. There is also a pipe at the bottom of the reservoir that allows the liquid to flow out of the reservoir; this corresponds to one's ability to dissipate negative affect.

When the input (inflow) and dissipation (outflow) rates are approximately balanced, the trajectory appears very much like someone who is self-regulating (Fig. 8, top row). If one's dissipation rate is larger than the rate of input, however,

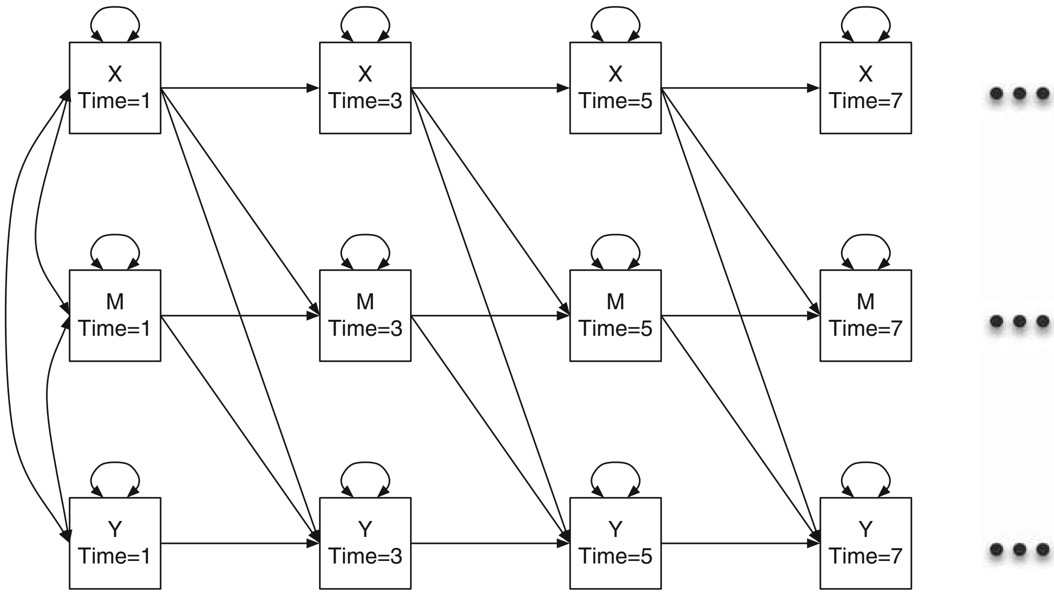

Fig. 9 A cross-lagged panel model

there is a tendency for floor effects to occur (Fig. 8, bottom row).[2] This model is one example of when a single differential equation can produce time series that appear qualitatively different, or time series with differing distributions for the dependent variable. Such models may be useful for identifying important parameters for characterizing intraindividual variability. Initially, it will be important to show how the average perceived input and dissipation parameters differentially relate to traits in the literature, but eventually these parameters may help to parse out similarities and differences between traits as well as identify traits that have been overlooked.

5.3 *Model 3*

The presentation of Models 1 and 2 has focused on new questions related to characterizing intraindividual variability and the relationship between different levels of the same variable that can be addressed using differential equations. The language of differential equations also has the potential to give new insight into old problems. The cross-lagged panel model (CLPM) in Fig. 9 has often been used in applications of longitudinal mediation. The ability to make a causal inference with this model is often stronger, although arguably not complete, because of the

[2]Videos demonstrating the evolution of these systems have been posted on the web site of the first author.

ability to test the directionality of relations such as X_t to M_{t+1} versus M_t to X_{t+1}. One limitation of drawing inferences with the CLPM is that inferences are limited to the specific lag at which data are collected (Cole and Maxwell 2003; Gollob and Reichardt 1987, 1991). Consequently, how to go about collecting data such that one selects the "correct" lag becomes a thorny issue, as the "correct" lag for one effect may not be the "correct" lag for another effect, and differing lags may be required for variables to reach their maximal influence (Cole and Maxwell 2003).

The CLPM is a discrete time model, as time is only implicitly considered through the order of the observations, but never explicitly considered through the specification of the time between observations (Voelkle et al. 2012). An alternative way of modeling data similar to the CLPM is by specifying a differential equation model that describes the underlying process that is generating the data. With such a model it would be possible to estimate the expected value of each variable at all times across the duration of the study, as time is explicitly considered in such a model. The expected values of the variables are calculated by integrating the differential equation model from some time t to some later time $t+\delta$.

One differential equation model that has been implemented frequently across many literatures is the model

$$\frac{dx}{dt} = Ax + \epsilon, \tag{13}$$

where the key difference with respect to Eq. (12) is that the errors are no longer all positive. Rather, ϵ is usually replaced with a continuous-time process that generates independent, normally distributed observations when integrated over some period of time (see Voelkle et al. 2012, for details). While Eqs. (12) and (13) appear very similar, the change in the distribution of the stochastic errors ϵ results in very different interpretations of β and A; while β addresses only the decay to zero, A is related to both increases and decreases that return the system to its steady state (see Deboeck and Boker in press, for examples and more details). The model in Eq. (13) can be rewritten in matrix form:

$$\frac{d\mathbf{X}}{dt} = \mathbf{AX} + \epsilon, \tag{14}$$

so as to allow it to be fit to more than one variable at a time, as in the CLPM in Fig. 9.

One advantage to using a differential equation model in this context is that the estimated parameters (e.g., $\mathbf{A}$) are independent of lag; that is, they do not depend on the spacing between repeated observations. Moreover, these parameters can be used to solve for the expected model parameters for differing lags; the lags for which one solves are not limited to those measured in one's data, although one should be cautious about examining lags that extrapolate beyond one's data. Figure 10 shows an example of the results that can be produced using differential equation models (Deboeck and Preacher 2013); the values of the lines, for any particular lag, can be

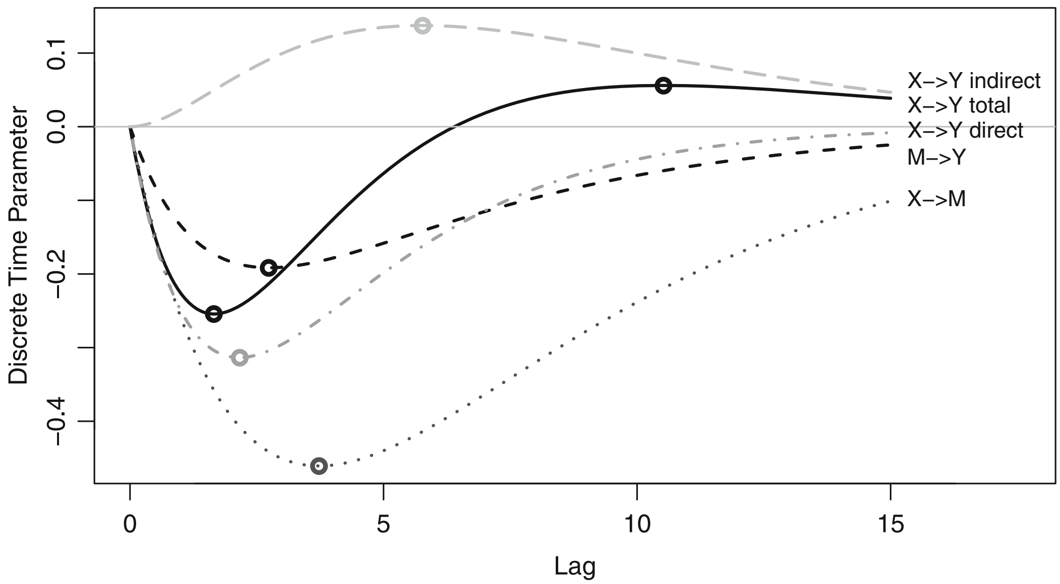

Fig. 10 Estimates based on differential equation model parameters of how the cross-lagged panel values (Discrete Time Parameters) change as a function of lag. The values of the lines, for any particular lag, can be interpreted as the parameters that would be expected if the cross-lagged panel model in Fig. 9 were fit to data with that particular lag. The maximal and minimal points have been marked with a *circle*, and the name of each effect is indicated on the *right*. The X to Y total effect represents the sum of the direct and indirect effects of X on Y

interpreted as the parameters that would be expected if the CLPM in Fig. 9 were fit to data with that particular lag assuming Eq. (14). Consequently, this figure shows how the discrete-time CLPM paths would be expected to change if data with differing lags were analyzed. While in a CLPM the results are typically presented for a single lag, corresponding to a single vertical slice through Fig. 10, with the differential equation model there is the potential to estimate relationships for many possible lags.

6 Concluding Remarks

In this chapter, the language of derivatives was introduced, and it was demonstrated how this language could be applied to familiar models to facilitate the appropriate application of these models to research questions related to change (e.g., HLM and LGCM). By using the language in Tables 1 and 2, precision in the specification of theories about change can be improved, methodology can be more easily identified, and accuracy of interpretation of results can be ensured. Perhaps most encouraging, this language framework also provides a structure within which new methods and models of change can be introduced, thus creating the potential to open new ways of formulating questions and ideas, such as those related to the analysis of intraindividual variability. Three models were explored to highlight some of the ways that dynamic, nonlinear, intraindividual variability can be characterized, and how these models have the potential to shed new light on old problems such as the dependency of results on sampling rate.

Curriculum used to train researchers on how to analyze research questions related to change already integrates some of the concepts presented in this chapter. Unfortunately, derivatives and differential equations are seldom presented in introductory texts, perhaps under the guise of simplifying the presentation of statistics. We propose this is a disservice to researchers as derivatives provide an appropriate framework to analyze change. Without training in the language of differential equation models, an incoherent framework may be presented for the different analytic approaches available for testing similar models. Without the Rosetta stone of derivatives, it is more difficult for researchers to integrate different approaches and systematically and effectively match their research question about change to the correct model. Rather than learning what makes models different, this approach first identifies the types of derivative relations present, and subsequently identifies key differences between models with differing names (i.e., LGCM versus LDE).

Differential equations have the potential to change the way we think about change, subsequently impacting the research questions asked and consequently the models fit. This is especially warranted given the increased focus on intraindividual variability that is occurring in many fields. Future decades promise to bring more application of statistics to individual lives, whether through personalized medicine, ecological momentary interventions, or other means. Learning the language of derivatives opens the floodgates to characterizing the rich complexity of intraindividual variability with interesting parameters that may be informative of unobserved processes. As with all languages, fluency takes practice; but fluency also provides a perspective, understanding, and beauty that is nearly unobtainable through translation.

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